Ordinal Sums of De Morgan Triples[†]

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ABSTRACT. In this paper we consider the ordinal sum of the summands of t-norms, t-conorms, and fuzzy negations and prove some new results about them. In particular, we provide some conditions in order to guarantee that when the summands are De Morgan triples their ordinal sums also are De Morgan triple. We also proved that De Morgan laws are preserved for the the action of automorphisms and *N*-duality.

Keywords: t-norm, t-conorm, fuzzy negation, De Morgan triples, ordinal sum.

1 INTRODUCTION

There are several ways to extend the propositional classical connectives for a set $[0,1]$, but not always these extentions preserve the properties of the classical conectives. Triangular norms (t-norms) and triangular conorms (t-conorms) were first studied by Menger [19] and, Scheweizer and Sklar [26] in probabilistic metric spaces in which triangular inequalities were extended using t-norms and t-conorms theory.

The name triangular norm refers to the fact that in the framework of probabilistic metric spaces t-norms are used to generalize triangle inequality of ordinary metric spaces and t-conorms are dual to t-norms under the order-reversing operation. The defining conditions of the t-norm are exactly those of the partially ordered Abelian monoid on the real unit interval [0, 1].

T-norms are used to represent logical conjunction in fuzzy logic and interseption in fuzzy set theory, whereas t-conorms are used to represent logical disjunction in fuzzy logic and union in fuzzy set theory. A t-norm (t-conorm) is a binary operation defined in $[0,1]$, which is commutative, associative, nondecreasing and with neutral element 1 (0).

In 19th century, De Morgan introduced the De Morgan's laws which in propositional logic and boolean algebra is a pair of transformation rules that are both valid rules of inference. This

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rules allow the expression of conjunctions and disjunctions purely in terms of each other via negation. In formal language, these rules can be expressed as "the negation of a conjunction is the disjunction of the negations" and "the negation of a disjunction is the conjunction of the negations". De Morgan's laws are also apply in the more general context of Boolean algebra and, in particular, in the Boolean algebra of set theory. In fuzzy logic, the triples formed by a t-norm, t-conorm and standard complement is called De Morgan triples if it fulfills De Morgan laws. De Morgan triples were first introduced by Zadeh in 1965.

In this contribuition, our aim is characterize the ordinal sum of the summands (a_i, b_i, N_i) where $(N_i)_{i \in I}$ are a family of fuzzy negations such that $N_i \geq N_S$ and $(]a_i, b_i[)_{i \in I}$ a family of nonempty, pairwise disjoint open subintervals of $[0,1]$, and prove that the function N is a fuzzy negation. In addition, we prove if (T_i, S_i, N_i) is a De Morgan triple satisfy some specific conditions, then (T, S, N) is a semi De Morgan triple.

This paper is organized as follows: Section 2 provides a review of the most important subjects for this work, such as t-norms, t-conorms, fuzzy negations, automorphisms and De Morgan triples. Section 3 contains some results based in [20], using definitions introduced in previous section. Section 4, we consider the notion of ordinal sum with respect to t-norms and t-conorms defined in [17] and characterize the ordinal sum with respect to fuzzy negations. In section 5, we provide some final remarks and furture works.

2   PRELIMINARIES

In this section, we will briefly review some basic concepts which are necessary for the development of this paper. The definitions and additional results can be found in [1], [2], [6], [4], [10], [9],[12], [15], [17].

2.1   t-norms, t-conorms and fuzzy negations

Definition 2.1. *A function* $T : [0,1]^2 \to [0,1]$ *is a t-norm if, for all* $x, y, z \in [0,1]$, *the following axioms are satisfied:*

1. *Commutative:* $T(x,y) = T(y,x)$;

2. *Associative:* $T(x, T(y,z)) = T(T(x,y), z)$;

3. *Monotonic: If* $x \leq y$, *then* $T(x,z) \leq T(y,z)$;

4. *One identity:* $T(x, 1) = x$.

A t-norm T is called positive if satifies the condiction: $T(x,y) = 0$ iff $x = 0$ or $y = 0$.

Example 2.1. *Some examples of t-norms:*

1. Gödel t-norm: $T_G(x,y) = \min(x,y)$;

2. Product t-norm: $T_P(x,y) = x.y;$

3. Lukasiewicz t-norm: $T_L(x,y) = \max(0, x+y-1);$

4. Drastic t-norm:

$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2; \\ \min(x,y) & \text{otherwise.} \end{cases}$$

$\square$

Definition 2.2. *A function* $S : [0,1]^2 \to [0,1]$ *is a t-conorm if, for all* $x,y,z \in [0,1]$, *the following axioms are satisfied:*

1.*Commutative:* $S(x,y) = S(y,x);$

2. *Associative:* $S(x,S(y,z)) = S(S(x,y),z);$

3. *Monotonic: If* $x \leq y$, *then* $S(z,x) \leq S(z,y);$

4. *Zero identity:* $S(x,0) = x.$

A t-conorm S is called positive if satifies the condiction: $S(x,y) = 1$ iff $x = 1$ or $y = 1$.

Example 2.2. *Some examples of t-conorms:*

1. Gödel t-conorm: $S_G(x,y) = \max(x,y);$

2. Probabilistic sum: $S_P(x,y) = x+y-x.y;$

3. Lukasiewicz t-conorm: $S_L(x,y) = \min(x+y,1);$

4. Drastic sum:

$$S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in]0,1]^2; \\ \max(x,y) & \text{otherwise.} \end{cases}$$

$\square$

A function $N : [0,1] \to [0,1]$ is a fuzzy negation if

N1: $N(0) = 1$ and $N(1) = 0;$

N2: Decreasing: If $x \leq y$, then $N(x) \geq N(y)$, for all $x,y \in [0,1]$.

A fuzzy negations N is strict if it is continuous and strictly decreasing, i.e., $N(x) < N(y)$ when $y < x$. A fuzzy negations N that satisfying the condition N3 is called strong.

N3: Involutive: $N(N(x)) = x$ for each $x \in [0, 1]$.

A fuzzy negation is called crisp if satisfies N4

N4: Crisp: For all $x \in [0, 1], N(x) \in \{0, 1\}$.

Example 2.3. *Some examples of fuzzy negations:*

1. Standard negation: $N_S(x) = 1 - x$;

2. Strict non-strong negation: $N_{S^2}(x) = 1 - x^2$;

3. Bottom negation:

$$N_\perp(x) = \begin{cases} 0 & if \quad x > 0 \\ 1 & if \quad x = 0 \end{cases}$$

4. Top negation:

$$N_\top(x) = \begin{cases} 0 & if \quad x = 1 \\ 1 & if \quad x < 1 \end{cases}$$

In [13], Dimuro *et.al.* define N_α and N^α as

$$N_\alpha(x) = \begin{cases} 0 & \text{if} \quad x > \alpha \\ 1 & \text{if} \quad x \le \alpha \end{cases} \tag{2.1}$$

$$N^\alpha(x) = \begin{cases} 0 & \text{if} \quad x \ge \alpha \\ 1 & \text{if} \quad x < \alpha \end{cases} \tag{2.2}$$

In [13] is proved that a fuzzy negation N is crisp iff there exists a $\alpha \in [0, 1]$ such that $N = N_\alpha$ or $N = N^\alpha$.

Definition 2.3. *[3, Definitions 2.3.8 and 2.3.14] Let T be a t-norm, S be a t-conorm and N be a fuzzy negation. The pair (T, N) satisfies the law of contradiction if*

$$T(x, N(x)) = 0, \ \forall \, x \in [0, 1] \tag{LC}$$

The pair (S, N) satisfies the law of excluded middle if

$$S(x, N(x)) = 1, \ \forall \, x \in [0, 1] \tag{LEM}$$

Note that:

1. If N is strong then it has an inverse N^{-1} which also is a strict fuzzy negation;

2. If N is strong then N is strict.

The supremum and infimum of the t-norms T_1 and T_2 is defined as

$$T_1 \wedge T_2(x,y) = \min\{T_1(x,y), T_2(x,y)\}$$
$$T_1 \vee T_2(x,y) = \max\{T_1(x,y), T_2(x,y)\}.$$

Similarly, we define the supremum and infimum of the t-conorms S_1 and S_2.

Proposition 2.1. *Let T_1 and T_2 be t-norms. Then, $T_1 \wedge T_2$ and $T_1 \vee T_2$ are symmetric, increasing and have 1 as neutral element.*

Proof. Let T_1 and T_2 be t-norms. Then, for all $x,y,z \in [0,1]$,

1. Symmetry:

$$
\begin{aligned}
T_1 \wedge T_2(x,y) &= \min\{T_1(x,y), T_2(x,y)\} \\
&= \min\{T_1(y,x), T_2(y,x)\} \\
&= T_1 \wedge T_2(y,x).
\end{aligned}
$$

2. Monotonicity: If $x \leq y$ then, since T is a t-norm, $T_1(x,z) \leq T_1(y,z)$ and $T_2(x,z) \leq T_2(y,z)$. Thus, $\min\{T_1(x,z), T_2(x,z)\} \leq \min\{T_1(y,z), T_2(y,z)\}$. Therefore, $T_1 \wedge T_2(x,z) \leq T_1 \wedge T_2(y,z)$.

3. Border Condition: For all $x \in [0,1]$, we have that

$$
\begin{aligned}
T_1 \wedge T_2(x,1) &= \min\{T_1(x,1), T_2(x,1)\} \\
&= \min\{x,x\} \\
&= x.
\end{aligned}
$$

Analogously we prove that $T_1 \vee T_2$ is symmetric, increasing and have 1 as neutral element.

Notice that, not always $T_1 \wedge T_2$ and $T_1 \vee T_2$ are associative and therefore, t-norms.

Proposition 2.2. *Let S_1 and S_2 be t-conorms. Then, $S_1 \wedge S_2$ and $S_1 \vee S_2$ are symmetric, increasing and have 0 as neutral element.*

Proof. Analogous from Proposition 2.1.

Definition 2.4. *Let T be a t-norm, S be a t-conorm and N be a strict fuzzy negation. T_N is the N-dual of T if, for all $x,y \in [0,1]$, $T_N(x,y) = N^{-1}(T(N(x),N(y)))$. Similarly, S_N is the N-dual of S if, for all $x,y \in [0,1]$, $S_N(x,y) = N^{-1}(S(N(x),N(y)))$.*

Proposition 2.3. *Let T be a t-norm, S be a t-conorm and N be a strict fuzzy negation. Then, T_N is a t-conorm and S_N is a t-norm.*

Proof. Let T be a t-norm and N be a fuzzy negation. Then, for all $x, y, z \in [0, 1]$,

1. Symmetry:

$$
\begin{aligned}
T_N(x,y) &= N^{-1}(T(N(x),N(y))) \\
&= N^{-1}(T(N(y),N(x))) \\
&= T_N(y,x).
\end{aligned}
$$

2. Associativity:

$$
\begin{aligned}
T_N(x, T_N(y,z)) &= N^{-1}(T(N(x),N(N^{-1}(T(N(y),N(z)))))) \\
&= N^{-1}(T(N(x),T(N(y),N(z)))) \\
&= N^{-1}(T(T(N(x),N(y)),N(z))) \\
&= N^{-1}(T(N(N^{-1}(T(N(x),N(y)))),N(z))) \\
&= T_N(T_N(x,y),z)
\end{aligned}
$$

3. Monotonicity: If $y \leq z$, then $N(z) \leq N(y)$. Since T is a t-norm, then $T(N(x),N(z)) \leq T(N(x),N(y))$. Thus, $N^{-1}(T(N(x),N(y))) \leq N^{-1}(T(N(x),N(z)))$. Therefore, $T_N(x,y) \leq T_N(x,z)$.

4. Border Condition: For all $x \in [0,1]$, we have that

$$
\begin{aligned}
T_N(x,0) &= N^{-1}(T(N(x),N(0))) \\
&= N^{-1}(T(N(x),1)) \\
&= N^{-1}(N(x)) \\
&= x.
\end{aligned}
$$

Therefore, T_N is a t-conorm. Analogously we prove that S_N is a t-norm.

Definition 2.5. *A function $\rho : [0,1] \to [0,1]$ is an automorphism if it is bijective and increasing, i.e., for each $x, y \in [0, 1]$, if $x \leq y$, then $\rho(x) \leq \rho(y)$.*

According with [9, Definition 0], a function $\rho : U \to U$ is an **automorphism** if it is continuous, strictly increasing and verifies the boundary conditions $\rho(0)=0$ and $\rho(1)=1$, i.e., if it is an increasing bijection on U, meaning that for each $x, y \in [0, 1]$, if $x \leq y$, then $\rho(x) \leq \rho(y)$. The set of all automorphisms on $[0, 1]$ will be denoted by $Aut([0, 1])$.

Automorphisms are closed under composition, i.e., if $\rho, \rho' \in Aut([0,1])$, then $\rho \circ \rho' \in Aut([0,1])$, where $\rho \circ \rho'(x) = \rho(\rho'(x))$. In addition, the inverse ρ^{-1} of an automorphism ρ is also an automorphism.

The function T^{ρ} (S^{ρ}), called as the **ρ-conjugate** of a t-(co)norm T (S), is obtained by action of $\rho \in Aut([0,1])$ on T (S) and defined in the following:

$$
\begin{aligned}
T^{\rho}(x,y) &= \rho^{-1}(T(\rho(x),\rho(y))), & (2.3) \\
S^{\rho}(x,y) &= \rho^{-1}(S(\rho(x),\rho(y))), \forall \, x,y \in [0,1]. & (2.4)
\end{aligned}
$$

2.2 De Morgan triples

According to [17], the triple (T,S,N) where T is a t-norm, S is a t-conorm and N a fuzzy negation is called De Morgan triples if satisfies the following conditions:

$$
T(x,y) = N(S(N(x),N(y)));
$$
$$
S(x,y) = N(T(N(x),N(y))),
$$

which naturally imply that N is a strong fuzzy negation [12].

There are several different notions of De Morgan triples as we can see in [7], [11], [16], [18], [21], [22]. In this paper we will use the definition of De Morgan triples laws as done in [12], [18], [22], [29] which not implies in involution of the fuzzy negation. Thus,

Definition 2.6. *Let T is a t-norm, S a t-conorm, N a fuzzy negation. Then (T,S,N) is a De Morgan triple if, for each $x,y \in [0,1]$,*

$$
\begin{aligned}
N(T(x,y)) &= S(N(x),N(y)); & (2.5) \\
N(S(x,y)) &= T(N(x),N(y)). & (2.6)
\end{aligned}
$$

Example 2.4. *(T_G, S_G, N_S), $(T_P, S_P, N_{\perp})$ and (T_L, S_L, N_S) are examples of De Morgan triples.* □

Definition 2.7. *(T,S,N) is a semi De Morgan triple if satify the Eq. (2.5) for each $x \in [0,1]$ or (2.6) for each $x \in [0,1]$.*

3   RESULTS ABOUT DE MORGAN TRIPLES

In this section, we will prove some propositions using definitions introduced on the previous section. The results of this section is based in [20].

Proposition 3.4. *Let T be a t-norm, S be a t-conorm and $\alpha \in [0,1]$. If, $T(x,y) > \alpha \Leftrightarrow x,y > \alpha$ and $S(x,y) \le \alpha \Leftrightarrow x,y \le \alpha$, then (T,S,N_{α}) is a De Morgan triple.*

Proof. Suppose that $x,y > \alpha$, then $T(x,y) > \alpha$ and $N_\alpha(T(x,y)) = 0$. On the other hand, $S(N_\alpha(x),N_\alpha(y)) = S(0,0) = 0$;

If $x \leq \alpha$, then $T(x,y) \leq \alpha$ and therefore, $N_\alpha(T(x,y)) = 1$. On the other hand, $S(N_\alpha(x),N_\alpha(y)) = S(1,N_\alpha(y)) = 1$;

If $y \leq \alpha$, the proof is analogous. Therefore, $N_\alpha(T(x,y)) = S(N_\alpha(x),N_\alpha(y))$.

Now, we will prove that $N_\alpha(S(x,y)) = T(N_\alpha(x),N_\alpha(y))$. Suppose that $x,y \leq \alpha$, then $S(x,y) \leq \alpha$ and $N_\alpha(S(x,y)) = 1$. On the other hand, $T(N_\alpha(x),N_\alpha(y)) = T(1,1) = 1$;

If $x > \alpha$, then $S(x,y) > \alpha$ and therefore, $N_\alpha(S(x,y)) = 0$. On the other hand, $T(N_\alpha(x),N_\alpha(y)) = T(0,N_\alpha(y)) = 0$;

If $y > \alpha$, the proof is analogous. Therefore, $N_\alpha(S(x,y)) = T(N_\alpha(x),N_\alpha(y))$.

Therefore, (T,S,N_α) is a De Morgan triple.

Corollary 3.0. *Let T be a positive t-norm and S be a positive t-conorm. Then, $(T,S,N_\perp)$ is a De Morgan triple.*

Proof. Analogous from Proposition 3.4.

Proposition 3.5. *Let T be a t-norm, S be a t-conorm and $\alpha \in [0,1]$. If, $T(x,y) \geq \alpha \Leftrightarrow x,y \geq \alpha$ and $S(x,y) < \alpha \Leftrightarrow x,y < \alpha$, then (T,S,N^α) is a De Morgan triple.*

Proof. Analogous from Proposition 3.4.

Corollary 3.0. *Let T be a positive t-norm and S be a positive t-conorm. Then, $(T,S,N_\top)$ is a De Morgan triple.*

Proof. Analogous from Proposition 3.4.

Proposition 3.6. *Let (T,S,N) be a De Morgan triple. If N is strict then (T,S,N^{-1}) is also De Morgan triple.*

Proof. Let $x,y \in [0,1]$, then

$$
\begin{aligned}
N^{-1}(T(x,y)) &= N^{-1}(T(N(N^{-1}(x)),N(N^{-1}(y)))) \\
&= N^{-1}(N(S(N^{-1}(x),N^{-1}(y)))) \\
&= S(N^{-1}(x),N^{-1}(y)).
\end{aligned}
$$

Analogously we prove that $N^{-1}(S(x,y)) = T(N^{-1}(x),N^{-1}(y))$.

The following lemmas will give us important results for to prove the propositions envolving contradiction law and law of excluded middle.

Lemma 3.1. *Let T be a t-norm and N be a strict fuzzy negation. (T,N) satisfies (LC) iff (T,N^{-1}) satisfies (LC).*

Proof. Since N is strict then it has an inverse strict fuzzy negation N^{-1}.
($\Rightarrow$) Since (T,N) satisfies (LC), then $T(N(x),N^{-1}(N(x))) = 0$, for all $x \in [0,1]$. So, for all $x \in [0,1]$, $T(x,N(x)) = T(N(x),x) = T(N(x),N^{-1}(N(x))) = 0$.
($\Leftarrow$) Analogous.

Lemma 3.2. *Let S be a t-conorm and N be a strict fuzzy negation. (S,N) satisfies (LEM) iff (S,N^{-1}) satisfies (LEM).*

Proof. Analogous from Lemma 3.1.

Proposition 3.7. *Let (T,S,N) be a semi De Morgan triple with respect the Eq. (2.5) and N be a strict fuzzy negation. If (T,N) satisfies (LC) then (S,N) satisfies (LEM).*

Proof. From Lemma 3.1, we have $S(N^{-1}(x),x) = 1$, $\forall\, x \in [0,1]$. Since (T,S,N) satisfies Eq. (2.5), we obtain that $0 = N(S(N^{-1}(x),x)) = T(x,N(x))$ for all $x \in [0,1]$.

Proposition 3.8. *Let (T,S,N) be a semi De Morgan triple with respect the Eq. (2.6) and N be a strict fuzzy negation. If (S,N) satisfies (LEM) then (T,N) satisfies (LC).*

Proof. From Lemma 3.2, we have $T(x,N^{-1}(x)) = 0$, $\forall\, x \in [0,1]$. Since (T,S,N) satisfies Eq. (2.6), we obtain that $1 = N(T(x,N^{-1}(x))) = S(N(x),x)$ for all $x \in [0,1]$.

Corollary 3.0. *Let (T,S,N) be a De Morgan triple and N be a strict fuzzy negation. Then, (T,N) satisfies (LC) iff (S,N) satisfies (LEM).*

Proof. Straightforward from Propositions 3.7 and 3.8.

Proposition 3.9. *Let (T,S,N) be a De Morgan triple. If N is strict then $(S_N,T_N,N) = (T,S,N)$.*

Proof. Since N is strict, then $S_N(x,y) = N^{-1}(S(N(x),N(y))) = N^{-1}(N(T(x,y))) = T(x,y)$. Analogously we prove that $T_N(x,y) = S(x,y)$.

Proposition 3.10. *Let (T,S,N) be a De Morgan triple. If N is strict, then $(S_N,T_{N^{-1}},N)$ and $(S_{N^{-1}},T_N,N)$ are semi De Morgan triple.*

Proof. Let N be a strict fuzzy negation and $x,y \in [0,1]$. Then

$$
\begin{aligned}
N(S_N(x,y)) &= N(N^{-1}(S(N(x),N(y)))) \\
&= S(N(x),N(y)) \\
&= N(T(x,y)) \\
&= N(T(N^{-1}(N(x)),N^{-1}(N(y)))) \\
&= T_{N^{-1}}(N(x),N(y)).
\end{aligned}
$$

Thus, $(S_N,T_{N^{-1}},N)$ satisfies the Eq. (2.6) and therefore, it is a semi De Morgan triple. Analogously, we prove that $(S_{N^{-1}},T_N,N)$ satisfies Eq. (2.5).

Proposition 3.11. *Let (T_1, S_1, N) and (T_2, S_2, N) be De Morgan triples. Then, $T_1 \leq T_2$ iff $S_1 \geq S_2$.*

Proof. ($\Rightarrow$) Let $T_1 \leq T_2$ and $x, y \in [0, 1]$. Then, $N(S_1(x, y)) = T_1(N(x), N(y)) \leq T_2(N(x), N(y)) = N(S_2(x, y))$. Therefore, $S_1(x, y) \geq S_2(x, y)$.

($\Leftarrow$) Analogous.

Proposition 3.12. *Let (T_1, S_1, N) and (T_2, S_2, N) be De Morgan triples. If $T_1 \wedge T_2$ and $S_1 \vee S_2$ are t-norm and t-conorm, respectively, then $(T_1 \wedge T_2, S_1 \vee S_2, N)$ is a De Morgan triple. Dually, if $T_1 \vee T_2$ and $S_1 \wedge S_2$ are t-norm, t-conorm, respectively, then $(T_1 \vee T_2, S_1 \wedge S_2, N)$ is a De Morgan triple.*

Proof. Let $x, y \in [0, 1]$. Then,

$$
\begin{aligned}
N(T_1 \wedge T_2(x, y)) &= N(\min\{T_1(x, y), T_2(x, y)\}) \\
&= \max\{N(T_1(x, y)), N(T_2(x, y))\} \\
&= \max\{S_1(N(x), N(y)), S_2(N(x), N(y))\} \\
&= S_1 \vee S_2(N(x), N(y)).
\end{aligned}
$$

Analogously we prove that $N(S_1 \vee S_2(x, y)) = T_1 \wedge T_2(N(x), N(y))$. Therefore, $(T_1 \wedge T_2, S_1 \vee S_2, N)$ is a De Morgan triple.

Dually, we prove that $(T_1 \vee T_2, S_1 \wedge S_2, N)$ is a De Morgan triple.

Proposition 3.13. *Let (T, S, N) be a De Morgan triple. If N is strict then $T = S_N = S_{N^{-1}}$ and $S = T_N = T_{N^{-1}}$.*

Proof. Since N is strict, then $S_N(x, y) = N^{-1}(S(N(x), N(y))) = N^{-1}(N(T(x, y))) = T(x, y)$ and $T(x, y) = T(N(N^{-1}(x)), N(N^{-1}(y))) = N(S(N^{-1}(x), N^{-1}(y))) = S_{N^{-1}}(x, y)$. Therefore, $T = S_N = S_{N^{-1}}$. Analogously we prove that $S = T_N = T_{N^{-1}}$.

Now, using the notion of automorphism ρ from t-norms, t-conorms and fuzzy negation, we show that the triple (T^ρ, S^ρ, N^ρ) is a De Morgan triple.

Proposition 3.14. *Let (T, S, N) be a De Morgan triple and ρ be an automorphism. Then, (T^ρ, S^ρ, N^ρ) is a De Morgan triple.*

Proof. Let $x, y \in [0, 1]$, then

$$
\begin{aligned}
N^\rho(T^\rho(x, y)) &= N^\rho(\rho^{-1}(T(\rho(x), \rho(y)))) \\
&= \rho^{-1}(N(\rho(\rho^{-1}(T(\rho(x), \rho(y)))))) \\
&= \rho^{-1}(N(T(\rho(x), \rho(y)))) \\
&= \rho^{-1}(S(N(\rho(x)), N(\rho(y)))) \\
&= \rho^{-1}(S(\rho(\rho^{-1}(N(\rho(x)))), \rho(\rho^{-1}(N(\rho(y)))))) \\
&= S^\rho(\rho^{-1}(N(\rho(x))), \rho^{-1}(N(\rho(y)))) \\
&= S^\rho(N^\rho(x), N^\rho(y)).
\end{aligned}
$$

Analogously we prove that $N^\rho(S^\rho(x,y)) = T^\rho(N^\rho(x), N^\rho(y))$.

4   ORDINAL SUMS OF DE MORGAN TRIPLES

In this section, we consider the notion of ordinal sum with respect to t-norms and t-conorms defined in [17] and characterize the ordinal sum with respect to fuzzy negations.

Proposition 4.15. *[17] Let $(T_i)_{i \in I}$ be a family of t-norms and $(]a_i, b_i[)_{i \in I}$ be a family of nonempty, pairwise disjoint open subintervals of $[0,1]$. Then the function $T : [0,1]^2 \to [0,1]$ defined by*

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) \, T_i \left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i} \right) & \text{if } (x,y) \in [a_i, b_i]^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{4.1}$$

is a t-norm which is called the ordinal sum of the summands (a_i, b_i, T_i), $i \in I$.

Proposition 4.16. *[17] Let $(S_i)_{i \in I}$ be a family of t-conorms and $(]a_i, b_i[)_{i \in I}$ be a family of nonempty, pairwise disjoint open subintervals of $[0,1]$. Then the function $S : [0,1]^2 \to [0,1]$ defined by*

$$S(x,y) = \begin{cases} a_i + (b_i - a_i) \, S_i \left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i} \right) & \text{if } (x,y) \in [a_i, b_i]^2, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{4.2}$$

is a t-conorm which is called the ordinal sum of the summands (a_i, b_i, S_i), $i \in I$.

Definition 4.8. *Let $(N_i)_{i \in I}$ be a family of fuzzy negations and $(]a_i, b_i[)_{i \in I}$ be a family of nonempty, pairwise disjoint open subintervals of $[0,1]$. Then the function $N : [0,1] \to [0,1]$ defined by*

$$N(x) = \begin{cases} (1 - b_i) + (b_i - a_i) \, N_i \left(\dfrac{x - a_i}{b_i - a_i} \right) & \text{if } x \in [a_i, b_i], \\ N_S(x) & \text{otherwise.} \end{cases} \tag{4.3}$$

is called of the ordinal sum of the summands (a_i, b_i, N_i), $i \in I$.

Lemma 4.3. *Let $(]a_i, b_i[)_{i \in I}$ be a family of nonempty, pairwise disjoint open subintervals of $[0,1]$, $(N_i)_{i \in I}$ be a family of fuzzy negations and N the ordinal sum N of the summands (a_i, b_i, N_i), $i \in I$. Then,*

1. *If $x \in [a_i, b_i]$ for some $i \in I$ then $N(x) \in [1 - b_i, 1 - a_i]$.*

2. *If $x \notin \bigcup_{i \in I} [a_i, b_i]$ then $N(x) \notin \bigcup_{i \in I} [1 - b_i, 1 - a_i]$.*

Proof. If $x \in [a_i, b_i]$ for some $i \in I$ then by Eq. (4.3) and because N_i is a fuzzy negation, we have that

$$N(x) \in \left[(1 - b_i) + (b_i - a_i) \, N_i \left(\frac{b_i - a_i}{b_i - a_i} \right), (1 - b_i) + (b_i - a_i) N_i \left(\frac{a_i - a_i}{b_i - a_i} \right) \right]$$
$$= \quad [(1 - b_i) + (b_i - a_i) \, N_i(1), (1 - b_i) + (b_i - a_i) \, N_i(0)] = [1 - b_i, 1 - a_i].$$

If $x \notin \bigcup_{i \in I}[a_i, b_i]$ then by Eq. (4.3), $N(x) = 1 - x$. Suppose, that $1 - x \in [1 - b_i, 1 - a_i]$ for some $i \in I$, then, trivially, $x \in [a_i, b_i]$ which is a contradiction. Therefore, $N(x) \notin \bigcup_{i \in I}[1 - b_i, 1 - a_i]$.

Proposition 4.17. *Let* $(]a_i, b_i[)_{i \in I}$ *be a family of nonempty, pairwise disjoint open subintervals of* $[0, 1]$ *and* $(N_i)_{i \in I}$ *be a family of fuzzy negations. Then the ordinal sum N of the summands* (a_i, b_i, N_i), $i \in I$ *is a fuzzy negation.*

Proof. If there exists $i \in I$ such that $a_i = 0$, then by Eq. (4.3), $N(0) = (1 - b_i) + b_i N_i(0) = 1$. On the other hand, if there exists no $i \in I$ such that $a_i = 0$, then by Eq. (4.3), $N(0) = N_S(0) = 1$. In analogous way, we can prove that $N(1) = 0$.

If $x \leq y$ then we have the following cases:

Case 1: If $x, y \in [a_i, b_i]$ for some $i \in I$, then $\dfrac{x - a_i}{b_i - a_i} \leq \dfrac{y - a_i}{b_i - a_i}$ and therefore $N_i\left(\dfrac{y - a_i}{b_i - a_i}\right) \leq N_i\left(\dfrac{x - a_i}{b_i - a_i}\right)$. So, Eq. (4.3), $N(y) \leq N(x)$.

Case 2: If $x \in [a_i, b_i]$ and $y \in [a_j, b_j]$ for some $i, j \in I$ such that $i \neq j$ then $a_i < b_i < a_j < b_j$. So, by Lemma 4.3, $N(y) \in [1 - a_j, 1 - b_j]$ and $N(x) \in [1 - b_i, 1 - a_i]$. Thus, since $1 - b_j < 1 - a_i$, then $N(y) < N(x)$.

Case 3: If $x \in [a_i, b_i]$ for some $i \in I$ and $y \notin \bigcup_{j \in I}[a_j, b_j]$, then $a_i \leq x$ and therefore $1 - y \leq 1 - a_i$. Since, by Lemma 4.3, $1 - a_i \leq N(x)$ and by Eq. (4.3) $N(y) = 1 - y$, then follows that $N(y) \leq N(x)$.

Case 4: If $x \notin \bigcup_{j \in I}[a_j, b_j]$ and $y \in [a_i, b_i]$ for some $i \in I$, then $x < b_i$. So, by Eq. (4.3) $N(x) = 1 - x$ and therefore, by Lemma 4.3, $N(y) \leq 1 - b_i \leq 1 - x = N(x)$.

Case 5: If $x, y \notin \bigcup_{i \in I}[a_i, b_i]$ then by Eq. (4.3), $N(y) = 1 - y \leq 1 - x = N(x)$.

Theorem 4.1. *Let* T, S *and* N *be the ordinal sum of the summands* (a_i, b_i, T_i), $(1 - b_i, 1 - a_i, S_i)$, (a_i, b_i, N_i), $i \in I$, *respectively. If, N is a fuzzy negation and for all* $i \in I$, (T_i, S_i, N_i) *is a De Morgan triple, then* (T, S, N) *is a semi De Morgan triple which satisfies Eq. (2.5).*

Proof. Let $x, y \in [0, 1]$. If $(x, y) \notin [a_i, b_i]^2$ for all $i \in I$, then by Lemma 4.3, $(N(x), N(y)) \notin [1 - b_i, 1 - a_i]^2$ for all $i \in I$. So, $N(T(x, y)) = N(\min(x, y)) = \max(N(x), N(y)) = S(N(x), N(y))$.

If $(x,y) \in [a_i, b_i]^2$ for some $i \in I$ then, $T(x,y) \in [a_i, b_i]$ and by Propositions 4.15, 4.16 and 4.17, we have that

$$
\begin{aligned}
N(T(x,y)) &= (1-b_i) + (b_i - a_i)N_i\left(\frac{T(x,y) - a_i}{b_i - a_i}\right) \\
&= (1-b_i) + (b_i - a_i)N_i\left(\frac{a_i + (b_i - a_i)\,T_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right) - a_i}{b_i - a_i}\right) \\
&= (1-b_i) + (b_i - a_i)N_i\left(T_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right)\right) \\
&= (1-b_i) + (b_i - a_i)S_i\left(N_i\left(\frac{x - a_i}{b_i - a_i}\right), N_i\left(\frac{y - a_i}{b_i - a_i}\right)\right) \\
&= (1-b_i) + (b_i - a_i)S_i\left(\frac{N(x) - (1-b_i)}{b_i - a_i}, \frac{N(y) - (1-b_i)}{b_i - a_i}\right) \\
&= S(N(x), N(y)).
\end{aligned}
$$

Theorem 4.2. *Let T and S be the ordinal sum of the summands (a_i, b_i, T_i) and $(1 - b_i, 1 - a_i, S_i)$, $i \in I$, respectively. If, for all $i \in I$, (T_i, S_i, N_S) is a De Morgan triple, then (T, S, N_S) is a De Morgan triple.*

Proof. Since N_S is the ordinal sums of (a_i, b_i, N_S) then from Theorems 4.1 we have that $N_S(T(x,y)) = S(N_S(x), N_S(y))$.

If $(x,y) \notin [a_i, b_i]^2$ for all $i \in I$, then, for all $i \in I$, $(1 - x, 1 - y) \notin [1 - b_i, 1 - a_i]^2$ and therefore $N_S(S(x,y)) = N_S(\max(x,y)) = \min(1 - x, 1 - y) = T(N_S(x), N_S(y))$.

If $x, y \in [1 - b_i, 1 - a_i]$ for some $i \in I$ then $S(x,y) \in [1 - b_i, 1 - a_i]$ and by Propositions 4.15, 4.16 and 4.17 (observe that N_S also is the ordinal sums of $(1 - b_i, 1 - a_i, N_S)$), we have that

$$
\begin{aligned}
N_S(S(x,y)) &= a_i + (b_i - a_i)N_S\left(\frac{S(x,y) - (1 - b_i)}{b_i - a_i}\right) \\
&= a_i + (b_i - a_i)N_S\left(\frac{(1 - b_i) + (b_i - a_i)\,S_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right) - (1 - b_i)}{b_i - a_i}\right) \\
&= a_i + (b_i - a_i)N_S\left(S_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right)\right) \\
&= a_i + (b_i - a_i)T_i\left(N_S\left(\frac{x - a_i}{b_i - a_i}\right), N_S\left(\frac{y - a_i}{b_i - a_i}\right)\right) \\
&= a_i + (b_i - a_i)T_i\left(\frac{N_S(x) - (1 - b_i)}{b_i - a_i}, \frac{N_S(y) - (1 - b_i)}{b_i - a_i}\right) \\
&= T(N_S(x), N_S(y)).
\end{aligned}
$$

Theorem 4.3. *Let T, S and N be the ordinal sum of the summands $(1 - b_i, 1 - a_i), T_i)$, (a_i, b_i, S_i), (a_i, b_i, N_i), $i \in I$, respectively. If N is a fuzzy negation and, for all $i \in I$, (T_i, S_i, N_i) is a De Morgan triple, then (T, S, N) is a semi De Morgan triple which satisfies Eq. (2.6).*

Proof. Analogously from the Theorem 4.1.

Theorem 4.4. *Let T and S be the ordinal sum of the summands $(1 - b_i, 1 - a_i, T_i)$ and (a_i, b_i, S_i), $i \in I$, respectively. If, for all $i \in I$, (T_i, S_i, N_S) is a De Morgan triple, then (T, S, N_S) is a De Morgan triple.*

Proof. Analogously from the Theorem 4.2.

5   CONCLUSION AND FINAL REMARKS

This work is a extension of the paper [20] presented in Fourth Brazilian Conference on Fuzzy Systems (IV CBSF), 2016. In this paper we consider the ordinal sum of the summands of a family of t-norms and t-conorms, and characterize the ordinal sum of the summands (a_i, b_i, N_i) where $(N_i)_{i \in I}$ are a family of fuzzy negations and $(]a_i, b_i[)_{i \in I}$ a family of nonempty, pairwise disjoint open subintervals of $[0, 1]$. In additon, we prove that the ordinal sum of the summands (a_i, b_i, N_i) is a fuzzy negation. Finally, we prove if (T_i, S_i, N_i) is a De Morgan triple satisfy some specific conditions, then (T, S, N) is a semi De Morgan triple as well as if (T_i, S_i, N_S) is a De Morgan triple, then (T, S, N_S) is a De Morgan triple, where N_S is a standard negation.

There are several types or extensions of fuzzy sets theory [8], among them is the fuzzy sets where the membership degrees are intervals proposed in an independent way in [23, 28]. As a future work, we will study ordinal sums of interval-valued De Morgan triples in the light of the interval representation ideas proposed in [5, 24, 25] which was adapted for interval-valued t-norms, t-conorms and fuzzy negations in [27, 6, 14]. In addition, we will also investigate additive and multiplicative generators of (interval-valued) De Morgan triples.

RESUMO. Neste artigo, consideramos a soma ordinal dos sumandos das t-normas, t-conormas e negações fuzzy, e provamos alguns novos resultados sobre eles. Em particular, fornecemos algumas condições no sentido de garantir que quando os somandos forem triplas de De Morgan suas somas ordinais também são triplas de De Morgan. Também provamos que as leis de De Morgan são preservadas pela ação de automorfismos e N-dualidade.

Palavras-chave: t-norma, t-conormas, negação fuzzy, triplas de De Morgan, soma ordinal.

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