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Applications of Nachbin's Theorem concerning Dense Subalgebras of Differentiable Functions

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ABSTRACT. In this paper, we give some applications of Nachbin's Theorem [4] to approximation and interpolation in the the space of all k times continuously differentiable real functions on any open subset of the Euclidean space.

Keywords: Nachbin's theorem, approximation of differentiable functions, Stone-Weierstrass theorem, interpolation.

1 INTRODUCTION

Let Ω be an open subset of $\mathscr{R}^p$ and let k be a nonnegative integer. We denote by $C^k(\Omega;\mathscr{R})$ the algebra of all k times continuously differentiable real functions on Ω and consider the compact open topology of order k τ_u^k, that is, the topology of uniform convergence for the functions and all their partial derivatives up to the order k on compact subsets of Ω.

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_p) \in \mathscr{N}_0^p$ of non-negative integers, let $|\alpha| := \alpha_1 + \cdots + \alpha_p$ be the order of α, $\alpha! := \alpha_1! \cdots \alpha_p!$, and for $|\alpha| \leq p$ let $D^\alpha := \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}$ represents the corresponding linear partial differential operator acting on $C^k(\Omega;\mathscr{R})$.

The topology τ_u^k is generated by the semi-norms $\sigma_{k,\Gamma}$ given by

$$\sigma_{k,\Gamma}(f) = \sum_{|\alpha|\leq k} \frac{1}{\alpha!} sup\{|(D^\alpha f)(x)| : x \in \Gamma\} \qquad \text{for all } f \in C^k(\Omega;\mathscr{R}),$$

where Γ runs over all compact subsets of Ω. By Proposition 3, p. 8 [5], $C^k(\Omega;\mathscr{R})$ is a topological vector space with respect to this topology.

In 1949 Nachbin [4] established the following interesting characterization of dense subalgebras of the space $C^k(\Omega;\mathscr{R})$.

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Theorem 1. *(Nachbin) Let Ω be an open subset of $\mathscr{R}^p$ and L be a subalgebra of $C^k(\Omega;\mathscr{R})$. Then L is dense in $C^k(\Omega;\mathscr{R})$ if and only if the following conditions are satisfied:*

(a) *given $x, y \in \Omega$ with $x \neq y$, there exists $f \in L$ such that $f(x) \neq f(y)$;*

(b) *given $x \in \Omega$, there exists $f \in L$ such that $f(x) \neq 0$;*

(c) *given $x \in \Omega$ and $u \in \mathscr{R}^p$ with $u \neq \mathbf{0}$, there exists $f \in L$ such that $\frac{\partial f}{\partial u}(x) \neq 0$.*

The proof of this result can be found in [3] and [4].

Our aim is to use Nachbin's theorem to give a proof of a density theorem and a simultaneous interpolation and approximation theorem in the space $C^k(\Omega;\mathscr{R})$.

2 THE RESULTS

The Urysohn's Lemma ([2] p. 281) for differentiable functions is the main tool we employed in the next lemma.

Lemma 1. *Let Ω be an open subset of $\mathscr{R}^p$, $w_1, \ldots, w_m$ distinct points in Ω, and $y_1, \ldots, y_m$ distinct real numbers. If L is a dense vector subspace of $C^k(\Omega;\mathscr{R})$, then there exists a function $h \in L$ such that $h(w_j) = y_j$, $j = 1, \ldots, m$.*

Proof. Let L be a dense linear subspace of $C^k(\Omega;\mathscr{R})$ and $S = \{w_1, \ldots, w_m\}$ be a subset of Ω. Consider the following linear mapping

$$\begin{array}{rcl} T : C^k(\Omega;\mathscr{R}) & \rightarrow & \mathscr{R}^m \\ f & \mapsto & (f(w_1), \ldots, f(w_m)). \end{array}$$

Notice that T is continuous.

For each $w_i \in S$ consider an open neighborhood $U_i \subset \Omega$ of w_i such that $w_j \notin U_i$, for all $j \neq i$, $j \in \{1, ..., m\}$. It follows from the Urysohn's Lemma for differentiable functions that there exists an infinitely differentiable function $\Phi_i : \mathscr{R}^m \rightarrow R$, $0 \leq \Phi_i \leq 1$, such that $\Phi_i(w_i) = 1$ and $\Phi_i(x) = 0$, if $x \notin U_i$, in particular, $\Phi_i(w_j) = 0$, $j \neq i$. Let $\phi_i = \Phi_i|_\Omega$ the restriction of the function Φ_i to the subset Ω and $e_i \in \mathscr{R}^m$ the vector whose i^{th} coordinate is equal to 1 and the others are equal to 0.

The linear mapping T is surjective since for any $(c_1, \ldots, c_m) \in \mathscr{R}^m$, we have

$$(c_1, \ldots, c_m) = \sum_{i=1}^{m} c_i e_i = \sum_{i=1}^{m} c_i(\phi_i(w_1), \ldots, \phi_i(w_m)) = \sum_{i=1}^{m} c_i T(\phi_i) = T\left(\sum_{i=1}^{m} c_i \phi_i\right),$$

where $\sum_{i=1}^{m} c_i \phi_i \in C^k(\Omega;\mathscr{R})$. Moreover, $T(L)$ is closed because it is a linear subspace of $\mathscr{R}^m$. Then by density of L and continuity of T, it follows that

$$\mathscr{R}^m = T(C^k(\Omega;\mathscr{R})) = T(\overline{L}) \subset \overline{T(L)} = T(L). \tag{2.1}$$

Therefore, for any $(y_1,\ldots,y_m)\in\mathscr{R}^m$ there exists $h\in L$ such that $T(h)=(y_1,\ldots,y_m)$, that is, $(h(w_1),\ldots,h(w_m))=(y_1,\ldots,y_m)$.

We give a proof of the following density result.

Theorem 2. *Let V be an open subset of $\mathscr{R}^p$, L a dense subalgebra of $C^k(V;\mathscr{R})$, and $v_1,\ldots,v_n$ distinct points in V. Consider the open subset of $\mathscr{R}^p$,*

$$\Omega = V\setminus\{v_1,\ldots,v_n\}$$

and the subalgebra

$$M=\{f|_\Omega : f\in L, f(v_1)=\ldots=f(v_n)=0\}.$$

Then, M is dense in $C^k(\Omega;\mathscr{R})$.

Proof. Clearly M is a subalgebra of $C^k(\Omega;\mathscr{R})$. Let x,y be any distinct points in Ω. Consider the following subset

$$S=\{x,y,v_1,\ldots,v_n\}$$

of V. By Lemma 1 there exists $h\in L$ such that $h(x)=1$, $h(y)=-1$ and $h(v_j)=0$ for $j=1,\ldots,n$. Then, $h|_\Omega\in M$ and satisfies Conditions (a) and (b) of Theorem 1.

Now let $z\in\Omega$ and $u\in\mathscr{R}^p$, $u\neq\mathbf{0}$. It follows from Lemma 1 that there exists $g\in L$ such that $g(z)=1$ and $g(v_j)=0$ for $j=1,\ldots,n$. Hence, $g|_\Omega\in M$. If $\frac{\partial g}{\partial u}(z)\neq 0$ the Condition (c) of Theorem 1 is satisfied. Otherwise, notice that L is not a subset of

$$B=\left\{f\in C^k(V;\mathscr{R}) : \frac{\partial f}{\partial u}(z)=0\right\},$$

since L is a dense subalgebra of $C^k(V;\mathscr{R})$ and B is a proper closed subalgebra of $C^k(V;\mathscr{R})$. Thus, there exists $\phi\in L$ such that $\frac{\partial\phi}{\partial u}(z)\neq 0$. Then, $\phi g\in L$ and $\phi g(v_j)=0$ for $j=1,\ldots,n$, that is, $\phi g|_\Omega\in M$. Moreover,

$$\frac{\partial\phi g}{\partial u}(z)=\frac{\partial\phi}{\partial u}(z)g(z)+\phi(z)\frac{\partial g}{\partial u}(z)=\frac{\partial\phi}{\partial u}(z)g(z)=\frac{\partial\phi}{\partial u}(z)\neq 0.$$

Thus, by Theorem 1, M is dense in $C^k(\Omega;\mathscr{R})$.

For each positive integer l, $\mathscr{P}^l(\mathscr{R}^p,\mathscr{R})$ denotes the linear subspace of $C^k(\mathscr{R}^p;\mathscr{R})$ generated by the set of all functions of the form

$$p(x)=[\psi(x)]^l,\qquad x\in\mathscr{R}^p,$$

where $\psi\in(\mathscr{R}^p)^*$, the dual space of $\mathscr{R}^p$. The elements of $\mathscr{P}^l(\mathscr{R}^p,\mathscr{R})$ are called the **l-homogeneous continuous polynomials of finite type from $\mathscr{R}^p$ into $\mathscr{R}$**. The subspace of $C^k(\mathscr{R}^p;\mathscr{R})$ consisting of all functions of the form

$$p(x) = p_0 + \sum_{j=1}^{l} p_j(x), \qquad x \in \mathscr{R}^p$$

where $p_0 \in \mathscr{R}$, $p_j \in \mathscr{P}^j(\mathscr{R}^p,\mathscr{R})$, $j = 1,\dots,l$, $l \in \mathscr{N}$, is denoted by $\mathscr{P}(\mathscr{R}^p,\mathscr{R})$. Its elements are called **real continuous polynomials of finite type**. The polarization formula shows that $\mathscr{P}(\mathscr{R}^p,\mathscr{R})$ is a subalgebra of $C^k(\mathscr{R}^p;\mathscr{R})$. Indeed, given ψ_1 and ψ_2 in $(\mathscr{R}^p)^*$,

$$\psi_1(x)\psi_2(x) = \frac{1}{4}[(\psi_1(x)+\psi_2(x))^2 - (\psi_1(x)-\psi_2(x))^2]$$

shows that $\psi_1\psi_2 \in \mathscr{P}^2(\mathscr{R}^p,\mathscr{R})$, since $\psi_1+\psi_2$ and $\psi_1-\psi_2$ belong to $(\mathscr{R}^p)^*$.

Corollary 3. *Let $v_1,\dots,v_n$ be distinct points in $\mathscr{R}^p$. Consider the open subset of $\mathscr{R}^p$,*

$$\Omega = \mathscr{R}^p \setminus \{v_1,\dots,v_n\}$$

and the subalgebra

$$M = \{f|_\Omega : f \in \mathscr{P}(\mathscr{R}^p;\mathscr{R}), f(v_1) = \dots = f(v_n) = 0\}.$$

Then, M is dense in $C^k(\Omega;\mathscr{R})$.

Proof. First of all, we verify that the subalgebra $P(\mathscr{R}^p;\mathscr{R})$ is dense in $C^k(\mathscr{R}^p;\mathscr{R})$. Given $x,y \in \mathscr{R}^p$ with $x \neq y$, it follows from Hahn-Banach Theorem that there exists $\psi \in (\mathscr{R}^p)^*$ such that $\psi(x) \neq \psi(y)$. Since $(\mathscr{R}^p)^* = P^1(\mathscr{R}^p;\mathscr{R}) \subset P(\mathscr{R}^p;\mathscr{R})$, the Condition (a) of Theorem 1 is satisfied. By definition, $P(\mathscr{R}^p;\mathscr{R})$ contains all the constant functions. Now, let $\mathbf{0} \neq u = (u_1,\dots,u_p) \in \mathscr{R}^p$. Then, there exists $0 \neq u_j \in \mathscr{R}$, $j \in \{1,\dots,p\}$. Let $\Pi_j : \mathscr{R}^p \to \mathscr{R}$ defined by $\Pi_j(x) = x_j$, $x \in \mathscr{R}^p$. Since $\frac{\partial \Pi_j}{\partial x_j}(x) = 1$ and $\frac{\partial \Pi_j}{\partial x_i}(x) = 0$ for $i \neq j$, it follows that

$$\frac{\partial \Pi_j}{\partial u}(x) = \sum_{i=1}^{p} u_i \frac{\partial \Pi_j}{\partial x_i}(x) = u_j \neq 0.$$

Therefore, by Theorem 1, $P(\mathscr{R}^p;\mathscr{R})$ is dense in $C^k(\mathscr{R}^p;\mathscr{R})$ and the assertion follows from Theorem 2.

Motivated by an extended Stone-Weierstrass theorem (see Corollary 1.1 [1]), we give a proof of a result concerning simultaneous interpolation and approximation in $C^k(\Omega;\mathscr{R})$. The tools are the Nachbin's Theorem and the following result due to Deutsch.

Theorem 4. *(Deutsch) Let Y be a dense vector subspace of the topological vector space Z and let $T_1,\dots,T_n$ be continuous linear functionals on Z. Then for each $f \in Z$ and each neighborhood U of f there is $y \in Y$ such that $y \in U$ and $T_i(y) = T_i(f)$, $i = 1,\dots,n$.*

Theorem 5. *Let Ω be an open subset of $\mathscr{R}^p$, $x_1,\dots,x_n$ distinct elements of Ω and L a subalgebra of $C^k(\Omega;\mathscr{R})$ that satisfies the following conditions,*

(a) *given* $x,y \in \Omega$ *with* $x \neq y$, *there exists* $f \in L$ *such that* $f(x) \neq f(y)$;

(b) *given* $x \in \Omega$, *there exists* $f \in L$ *such that* $f(x) \neq 0$;

(c) *given* $x \in \Omega$ *and* $u \in \mathscr{R}^p$ *with* $u \neq \mathbf{0}$, *there exists* $f \in L$ *such that* $\frac{\partial f}{\partial u}(x) \neq 0$.

Then, for each $f \in C^k(\Omega;\mathscr{R})$, *and each neighborhood* U *of* f *there exists* $g \in L \cap U$ *such that* $f(x_i) = g(x_i)$ *for* $i = 1,\ldots,n$.

Proof. It follows from Theorem 1 that L is a dense subalgebra of the topological vector space $C^k(\Omega;\mathscr{R})$. Let $S = \{x_1,...,x_n\} \subset \Omega$. Notice that

$$\begin{array}{rcl} T_i : C^k(\Omega;\mathscr{R}) & \to & \mathscr{R} \\ f & \mapsto & f(x_i) \end{array}$$

is a continuous linear functional for each $i = 1,\cdots,n$. Setting $Z = C^k(\Omega;\mathscr{R})$ and $Y = L$, the conclusion follows from Theorem 4.

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RESUMO. Em 1949, Leopoldo Nachbin estabeleceu uma versão do Teorema de Stone-Weierstrass para funções diferenciáveis de classe C^k em abertos do espaço euclidiano. Neste trabalho, apresentamos algumas aplicacões desse teorema relacionadas com aproximação e interpolação no espaço das funções de classe C^k munido da topologia compacto-aberta.

Palavras-chave: Teorema de Nachbin, aproximação de funções diferenciáveis, Teorema de Stone-Weierstrass, interpolação.

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