Tema
Tendências em Matemática Aplicada e Computacional, **21**, N. 1 (2020), 171-190

www.scielo.br/tema
doi: 10.5540/tema.2020.021.01.00171

An Extension of the Invariance Principle for Switched Affine System

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Received on December 11, 2018 / Accepted on November 19, 2019

ABSTRACT. In this paper, an approach to investigate switched affine system via matrix inequalities is presented. Particularly, an extension of LaSalle's invariance principle for this class of systems under arbitrary dwell-time switching signal is presented. The proposed results employ a common auxiliary scalar function and also multiple auxiliary scalar functions to study the asymptotic behavior of switched solutions and estimate their attractors for any dwell-time switching signal. A specific feature of these results is that the derivative of the auxiliary scalar functions can assume positive values in some bounded sets. Moreover, a problem of constrained optimization is formulated to numerically determine the auxiliary scalar functions and minimize the volume of the estimated attractor. Numerical examples show the potential of the theoretical results in providing information on the asymptotic behavior of solutions of the switched affine systems under arbitrary dwell-time switching signals.

Keywords: switched affine system, invariance principle, dwell-time, attractor set.

1 INTRODUCTION

Switched systems arise in practice when modeling the operation of many systems [9]. For this reason, important results about stability and stabilization for this class of system were presented in [2, 5, 8].

A subclass of nonlinear switched systems, known as switched affine system, can model some practical problems as well, especially in the area of electronics and power systems. An interesting application of this class of systems in electrical power systems can be found in [6]. Since these systems are subject to changes in the system equilibrium conditions due to fast varying loads, the focus in [6] was to determine conditions to ensure that the system trajectories remain confined into a security region of operation, even if the equilibrium point of the model changes.

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An important observation about the switched affine system is that its equilibrium points change according to the time switching signal. Therefore, in this paper we are not interested in studying the stability of a particular equilibrium point but the asymptotic behavior of solutions.

The invariance principle is a powerful tool to study the asymptotic behavior of dynamical system solutions. It was established for the class of nonlinear switched system in [2]. However, less conservative results were obtained considering the extension of LaSalle's invariance principle. The extension of the invariance principle was firstly obtained for continuous differential equations [14, 15] and afterwards for discrete systems [1], periodic systems [13] and switched nonlinear systems [17].

The invariance principle presented in [3] and [17] can be used to analyze the solutions of the switched affine system. However, the authors did not explore the particularities of the affine system to obtain the results. For this reason, in this paper, the properties of the affine system are explored to obtain sufficient conditions in terms of matrix inequalities to analyze the solution of this class of systems. More specifically, extensions of the invariance principle under a common auxiliary scalar function and also multiple auxiliary scalar functions will be presented. The main results are useful to estimate attractors of switched affine systems under arbitrary dwell-time switching signals.

From a practical point of view, the results proposed in this paper overcome the problem of finding the auxiliary scalar function and also the multiple auxiliary scalar functions satisfying all the conditions of the invariance principle presented in [17] and [3]. Moreover, the techniques that are used enabled us to construct a constrained optimization problem, which can numerically determine the auxiliary scalar function and the multiple auxiliary scalar functions, minimizing the volume of the estimated attractor. Preliminary results of this work were presented in [10] and [11]

The remainder of this paper is organized as follows. In Section 2, preliminary concepts of switched systems are presented; in Section 3, an extension of the invariance principle for arbitrary switched affine systems is presented; in Section 4, a systematic method to obtain optimal estimates of the attractor set of affine switched systems, which explores a nonlinear optimization problem, along with some numerical examples, is presented. Finally, the conclusion is presented in Section 5.

The notation used in this paper is fairly standard. Specifically, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}^n$ denotes the Euclidean space of dimension n and $\mathbb{R}^{n\times n}$ denotes the space of real matrices $n\times n$. The notation $\|\cdot\|$ refers to the Euclidean norm, $B(x,\varepsilon)$ denotes the open ball $\{y\in\mathbb{R}^n : \|y-x\| < \varepsilon\}$ radius ε centered in x and $B(\mathscr{M},\varepsilon)=\bigcup_{x\in\mathscr{M}} B(x,\varepsilon)$. The complement and boundary of set $\mathscr{M}$ is denoted by $\mathscr{M}^c$ and $\partial\mathscr{M}$ respectively. For matrices or vectors, $(')$ indicates transpose. In addition, for a matrix P, $P>0$ indicates that P is a real symmetric and a positive definite matrix and $\lambda_{\max}(P)$, $\lambda_{\min}(P)$ denote its minimum and maximum eigenvalue, respectively.

2 PRELIMINARIES

Consider the following class of switched system:

$$\dot{x} = f_{\sigma(t)}(x), \tag{2.1}$$

where $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathscr{C}^1$-function for all $p \in \mathscr{P} = \{1, \cdots, \mathscr{N}\}$, $\mathscr{N}$ is the number of subsystems, $x(t) \in \mathbb{R}^n$ is the state vector and $\sigma(t) : [0,\infty) \to \mathscr{P}$ is a piecewise constant function, continuous from the right, called switching signal. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of consecutive switching times associated with the switching signal σ and $I_p = \{t \in [\tau_k, \tau_{k+1}) : \sigma(\tau_k) = p, k \in \mathbb{N}\}$ be the union of intervals where subsystem p is active. The smooth, piecewise continuous function $x : I \to \mathbb{R}^n$ is a solution of the switched system (2.1) in the interval I if $x(t)$ satisfies $\dot{x}(t) = f_{\sigma(t)}(x(t))$, $\forall t \in I_p \cap I$ for all $p \in \mathscr{P}$. We assume that the sequence of switching times $\{\tau_k\}_{k\in\mathbb{N}}$ is divergent and that each subsystem p is active infinite times. The set of all switching solutions is denoted by $\mathscr{S}$. We denote $\varphi_{\sigma(t)}(t,x_0)$, the solution of the switched system (3.1) with initial condition x_0 at the time $t = 0$ under switching signal $\sigma(t)$.

Some preliminary definitions, which can be found in [7] and [3], are presented below for the switched system (2.1).

Definition 2.1. *The solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}$ has a non-vanishing dwell-time if there exists $h > 0$ so that $\inf_k(\tau_{k-1} - \tau_k) \geq h$ where $\{\tau_k\}_{k\in\mathbb{N}}$ is the sequence of consecutive switching times associated with $\varphi_{\sigma(t)}(t,x_0)$. The number h is called a dwell-time for $\varphi_{\sigma(t)}(t,x_0)$ and the set of all solutions possessing a non-vanishing dwell-time is denoted by $\mathscr{S}_{dwell} \subset \mathscr{S}$.*

Definition 2.2. *A point $q \in \mathbb{R}^n$ is a limit point of the continuous curve $\varphi_{\sigma(t)}(t,x_0) : [0,\infty) \to \mathbb{R}^n$ if there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$, with $t_k \to +\infty$ as $k \to +\infty$, so that $\lim_{k\to+\infty} \varphi_{\sigma(t_k)}(t_k,x_0) = q$. The set of all limit points of $\varphi_{\sigma(t)}(t,x_0)$ is denoted by $\omega_\sigma^+(x_0)$.*

The set $\omega_\sigma^+(x_0)$ of $\varphi_{\sigma(t)}(t,x_0)$ depends not only on the initial condition x_0 but also on the switching signal σ.

Definition 2.3. *The solution $\varphi_{\sigma(t)}(t,x_0) : [0,\infty) \to \mathbb{R}^n$ of (2.1) is attracted to a compact set $\mathscr{M}$ if for all $\varepsilon > 0$ there exists a time $\bar{t} > 0$ so that $\varphi_{\sigma(t)}(t,x_0) \in B(\mathscr{M},\varepsilon)$ for $t \geq \bar{t}$. Clearly, $\varphi_{\sigma(t)}(t,x_0)$ is attracted to a set $\mathscr{M}$, that is, $\varphi_{\sigma(t)}(t,x_0) \to \mathscr{M}$, if, and only if, $\lim_{t\to\infty} d(\varphi_{\sigma(t)}(t,x_0),\mathscr{M}) = 0$, where d is the distance between a point and a set, which is defined by $d(y,\mathscr{M}) = \inf_{m\in\mathscr{M}} \|y - m\|$.*

Definition 2.4. *A compact set $\mathscr{M}$ is weakly invariant in regard to the switched system (2.1) if for each $x_0 \in \mathscr{M}$, there exists an index $p \in \mathscr{P}$ and a real number $c > 0$ so that $\varphi_p(t,x_0) \in \mathscr{M}$ for any $t \in [-c,0]$ or $t \in [0,c]$.*

The following proposition, which is proven in [3], establishes properties of the limit set $\omega_\sigma^+(x_0)$ of bounded solutions

Proposition 2.1. *Let $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ be a bounded solution of (2.1) for $t \geq 0$. Then, $\omega_\sigma^+(x_0)$ is nonempty, compact and weakly invariant. Moreover, $\varphi_{\sigma(t)}(t,x_0)$ is attracted to $\omega_\sigma^+(x_0)$.*

In the next section, an extension of LaSalle's invariance principle for a subclass of switched systems (2.1) is proposed. This extension is useful for obtaining estimates of global attractor sets of switched affine systems.

3 AN INVARIANCE PRINCIPLE FOR SWITCHED AFFINE SYSTEMS

The purpose of this section is to analyze the asymptotic behavior of the solutions of the class of continuous-time affine switched systems

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}, \quad x(0) = x_0, \tag{3.1}$$

where $A_p \in \mathbb{R}^{n\times n}$, $b_p \in \mathbb{R}^n$, $\forall p \in \mathscr{P}$ and $\sigma(t)$ is a dwell-time switching signal, using an auxiliary common scalar function for all subsystems of the switched system (3.1) and multiple auxiliary scalar functions.

3.1 Results obtained via common auxiliary scalar functions

Consider a scalar quadratic function $V : \mathbb{R}^n \to \mathbb{R}$, which in the course of this text will be called an auxiliary function, given by

$$V(x) = (x-d)'P(x-d), \text{ where } P \in \mathbb{R}^{n\times n} \text{ and } d \in \mathbb{R}^n. \tag{3.2}$$

In addition, suppose that

$$\exists \ P > 0 \quad \text{satisfying} \quad Q_p = A_p'P + PA_p < 0, \ \ \forall p \in \mathscr{P}. \tag{3.3}$$

Now, let $\mathscr{D}_p = \{x \in \mathbb{R}^n : \nabla V(x)(A_p x + b_p) \geq 0\}$ be the set where the derivative of the auxiliary function V along the trajectories of the subsystem p is positive or null and $\mathscr{D} = \bigcup_{p\in\mathscr{P}} \mathscr{D}_p$. Let $\Omega_\ell^{P,d} = \{x \in \mathbb{R}^n : V(x) \leq \ell, \text{ where } \ell \in \mathbb{R}\}$ be a sublevel set of the auxiliary function (3.2) for a given P and d.

Lemma 1 provides sufficient conditions for the set $\mathscr{D}$ to be bounded by a sublevel of the auxiliary function V.

Lemma 1. *Consider the switched affine system (3.1) and the auxiliary function (3.2) such that (3.3) is satisfied. Then, the set $\mathscr{D}$ is bounded and there exists a real number*

$$\ell > \lambda_{\max}(P)\,(z + \|d\|)^2, \tag{3.4}$$

with $z = \max_{p\in\mathscr{P}} \left\{ -\frac{\mu_p + \sqrt{\mu_p^2 - 2\lambda_{\max}(Q_p)\xi_p}}{\lambda_{\max}(Q_p)} \right\}$, $\mu_p = \left\|b_p'P - d'PA_p\right\|$ *and* $\xi_p = \left|d'Pb_p\right|$, *which ensures the inclusion* $\mathscr{D} \subset \Omega_\ell^{P,d}$.

Proof. The derivative of the function V along the solution of subsystem p satisfies

$$\begin{aligned} \nabla V(x)(A_p x + b_p) &= x'Q_p x + 2(b_p'P - d'PA_p)x - 2d'Pb_p \\ &\leq x'\lambda_{\max}(Q_p)x + 2\left\|b_p'P - d'PA_p\right\|\|x\| + 2\left|d'Pb_p\right| \\ &= \lambda_{\max}(Q_p)\|x\|^2 + 2\mu_p\|x\| + 2\xi_p, \end{aligned}$$

where $\mu_p = \left\| b_p' P - d' P A_p \right\|$ and $\xi_p = \left| d' P b_p \right|$. Thus, we conclude that

$$\nabla V(x)(A_p x + b_p) \leq \lambda_{\max}(Q_p) \|x\|^2 + 2\mu_p \|x\| + 2\xi_p. \tag{3.5}$$

Since (3.3) is satisfied for all $p \in \mathscr{P}$, we have that $\lambda_{\max}(Q_p) < 0$. Thus, from (3.5), we conclude that the derivative of function V is strictly negative when $\|x\| > -\frac{\mu_p + \sqrt{\mu_p^2 - 2\lambda_{\max}(Q_p)\xi_p}}{\lambda_{\max}(Q_p)}$.

Then, $\mathscr{D}_p \subseteq \left\{ x \in \mathbb{R}^n : 0 \leq \|x\| \leq -\frac{\mu_p + \sqrt{\mu_p^2 - 2\lambda_{\max}(Q_p)\xi_p}}{\lambda_{\max}(Q_p)} \right\}$ and $\mathscr{D} = \bigcup_{p \in \mathscr{P}} \mathscr{D}_p \subseteq \{x \in \mathbb{R}^n : 0 \leq \|x\| \leq z\}$, where $z = \max_{p \in \mathscr{P}} \left\{ -\frac{\mu_p + \sqrt{\mu_p^2 - 2\lambda_{\max}(Q_p)\xi_p}}{\lambda_{\max}(Q_p)} \right\}$. Therefore, the set $\mathscr{D}$ is bounded.

Analyzing the values that the function V assumes when $x \in \mathscr{D}$, we obtain:

$$V(x) \leq \lambda_{\max}(P) \|x - d\|^2 \leq \lambda_{\max}(P) (\|x\| + \|d\|)^2 \leq \lambda_{\max}(P) (z + \|d\|)^2, \forall x \in \mathscr{D}.$$

Then, choosing $\ell \in \mathbb{R}$ such that $\ell > \lambda_{\max}(P)(z + \|d\|)^2$, we conclude that $\mathscr{D} \subset \Omega_\ell^{P,d}$. □

The next lemma guarantees the existence of a positively invariant set for the switched affine systems under an arbitrary dwell-time switching signal.

Lemma 2. *Consider the switched affine system (3.1) and the auxiliary function (3.2) such that (3.3) is satisfied. Let $\ell \in \mathbb{R}$ be a real number satisfying (3.4). If $x_0 \in \Omega_\ell^{P,d}$, then every solution $\varphi_{\sigma(t)}(t, x_0) \in \mathscr{S}_{dwell}$ with $x_0 \in \Omega_\ell^{P,d}$ stays inside $\Omega_\ell^{P,d}$ for all $t \geq 0$.*

Proof. For $x_0 \in \Omega_\ell^{P,d}$, let $\varphi_{\sigma(t)}(t, x_0) \in \mathscr{S}_{dwell}$ be a solution of the switched system (3.1) under arbitrary dwell-time switching $\sigma(t)$. Suppose the existence of $\bar{t} > 0$ so that $\varphi_{\sigma(\bar{t})}(\bar{t}, x_0) \notin \Omega_\ell^{P,d}$. Then, by the continuity of V and $\varphi_{\sigma(t)}(t, x_0)$, there exists $\tilde{t} \in (0, \bar{t})$ so that $V\left(\varphi_{\sigma(\tilde{t})}(\tilde{t}, x_0)\right) = \ell$ and $V(\varphi_{\sigma(t)}(t, x_0)) > \ell$, $\forall t \in (\tilde{t}, \bar{t}]$. Thus, V has to increase out of $\Omega_\ell^{P,d}$. On the other hand, according to Lemma 1, fixed the real number ℓ satisfying (3.4), $\mathscr{D} \subset \Omega_\ell^{P,d}$, which leads to a contradiction.

Therefore, the solution $\varphi_{\sigma(t)}(t, x_0) \in \mathscr{S}_{dwell}$ stays inside $\Omega_\ell^{P,d}$ for all $t \geq 0$ because every sublevel set of the function V is bounded. □

From Lemma 1 e Lemma 2, we can prove the following invariance principle for the class of switched affine systems using a common auxiliary function.

Theorem 3. *Consider the switched affine system (3.1) and the auxiliary function (3.2) such that (3.3) is satisfied. Then, every solution $\varphi_{\sigma(t)}(t, x_0) \in \mathscr{S}_{dwell}$ is attracted to a weakly invariant set in $\Omega_\ell^{P,d}$, where ℓ is given by (3.4).*

Proof. First, we consider $x_0 \in \Omega_\ell^{P,d}$, then, by Lemma 2 we have that every solution $\varphi_{\sigma(t)}(t, x_0) \in \mathscr{S}_{dwell}$ stays inside $\Omega_\ell^{P,d}$ for all $t \geq 0$, that is, the solution $\varphi_{\sigma(t)}(t, x_0) \in S_{dwell}$ is bounded. By Proposition 2.1 we conclude that the solution will be attracted to a weakly invariant set in $\Omega_\ell^{P,d}$.

Now, let $x_0 \notin \Omega_\ell^{P,d}$ and $\varphi_{\sigma(t)}(t, x_0) \in \mathscr{S}_{dwell}$. If $\varphi_{\sigma(t)}(t, x_0)$ enters $\Omega_\ell^{P,d}$ at some time t, then the result follows from the first part of this proof. Suppose the solution $\varphi_{\sigma(t)}(t, x_0) \notin \Omega_\ell^{P,d}$, $\forall t \geq 0$.

Since $\ell > \sup_{x\in\mathscr{D}} V(x)$, it follows that $\partial\Omega_\ell^{P,d} \cap \mathscr{D} = \emptyset$. This implies the existence of $\varepsilon > 0$ such that $\sup_{x\in\left(\Omega_\ell^{P,d}\right)^c} \nabla V(x)(A_p x + b_p) \leq -\varepsilon < 0$, $\forall p \in \mathscr{P}$. Therefore $V(\varphi_{\sigma(t)}(t,x_0))$ is strictly decreasing, which implies the existence of $\bar{t} \in \mathbb{R}$ such that $\varphi_{\sigma(\bar{t})}(\bar{t},x_0) \in \Omega_\ell^{P,d}$. By Lemma 2, the solution $\varphi_{\sigma(t)}(t,x_0) \in \Omega_\ell^{P,d}$ for all $t \geq \bar{t}$. Thus, the conclusion follows from the first part of this proof. Therefore, every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ is attracted to a weakly invariant set in $\Omega_\ell^{P,d}$. □

The following example illustrates the results of Theorem 3.

Example 3.1. *Consider the switched affine system*

$$\dot{x} = A_{\sigma(t)}x + b_{\sigma(t)}, \quad x \in \mathbb{R}^2, \tag{3.6}$$

where $\sigma(t) \in \mathscr{P} = \{1,2,3\}$ *and* $A_1 = \begin{bmatrix} -4 & 0 \\ 2 & -7 \end{bmatrix}$, $b_1 = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -5 \\ 2 & -6 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$, $A_3 = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$, $b_3 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. *The eigenvalues of the matrices* A_p, $p \in \{1,2,3\}$, *are* $\{-7,-4\}$, $\left\{-\frac{7}{2} \pm \frac{1921}{992}i\right\}$, $\{-3,-1\}$, *respectively. In addition, the equilibrium points of each subsystem p,* $p \in \{1,2,3\}$, *are different:* $x_{eq_1} = \left[-\frac{9}{4} \; -\frac{15}{14}\right]'$, $x_{eq_2} = \left[-\frac{9}{4} \; -\frac{1}{4}\right]'$ *and* $x_{eq_3} = [1 \; -2]'$.

With the objective of obtaining an estimate of the attractor set for the switched affine system (3.6), consider the auxiliary function (3.2), where P and d are given by $P = P_1 = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 4 \end{bmatrix}$ *and* $d = d_1 = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$, *respectively. Since* P_1 *satisfies (3.3), Lemma 1 ensures that the set* $\mathscr{D}$ *is bounded and* $\mathscr{D} \subset \Omega_{\bar{\ell}}^{P_1,d_1}$, *where* $\ell = \bar{\ell} = 419.5925$, *which satisfies (3.4). Then, from Theorem 3, every solution* $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ *is attracted to a weakly invariant set in* $\Omega_{\bar{\ell}}^{P_1,d_1}$. *Therefore, the attractor set of the system (3.6) is contained in the ellipsoidal region* $\Omega_{\bar{\ell}}^{P_1,d_1}$ *for any dwell-time switching. The volume of this estimation is vol* $\left(\Omega_{\bar{\ell}}^{P_1,d_1}\right) = 680.7098$.

Figure 1 illustrates $\Omega_{\bar{\ell}}^{P_1,d_1}$ *and a trajectory starting at* $x_0 = [-20 \;\; 20]'$ *under switching signal* $\sigma(t)$ *with dwell-time* $h = 0.2$ *seconds. This figure confirms the results of Theorem 3 by showing an attractor inside the set* $\Omega_{\bar{\ell}}^{P_1,d_1}$. *Function* $\dot{V}$ *along the switching solution is shown in Figure 2. Observe in Figure 2 the changes of sign of the derivate of V along the solution.*

3.2 Results obtained via multiple auxiliary scalar functions

Although Theorem 3 provides less conservative conditions on the auxiliary function V as compared to the LaSalle's invariance principle, it still may be difficult to find such V satisfying all assumptions of Theorem 3 for all $p \in \mathscr{P}$. Moreover, the function V can not exist, or it might lead

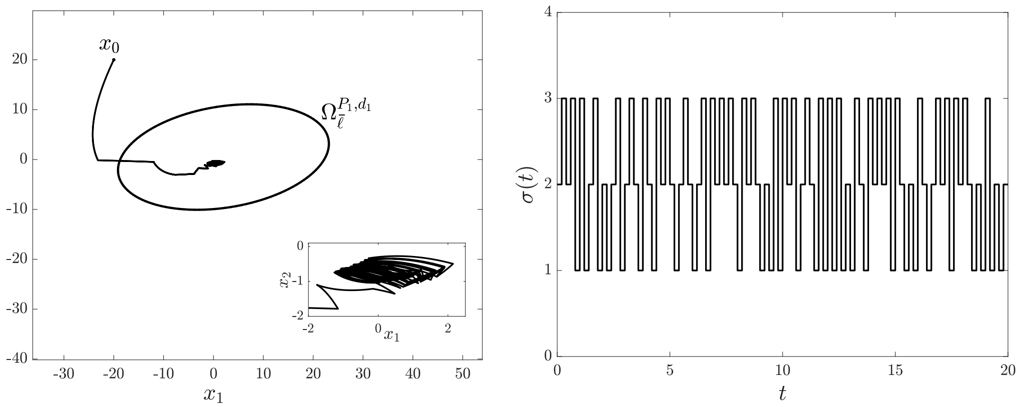

Figure 1: Phase portrait for Example 3.1 with initial condition $x_0 = [-20\ \ 20]'$ illustrating the level set $\Omega_{\bar{\ell}}^{P_1,d_1}$ and switching signal with dwell-time $h = 0.2$ seconds.

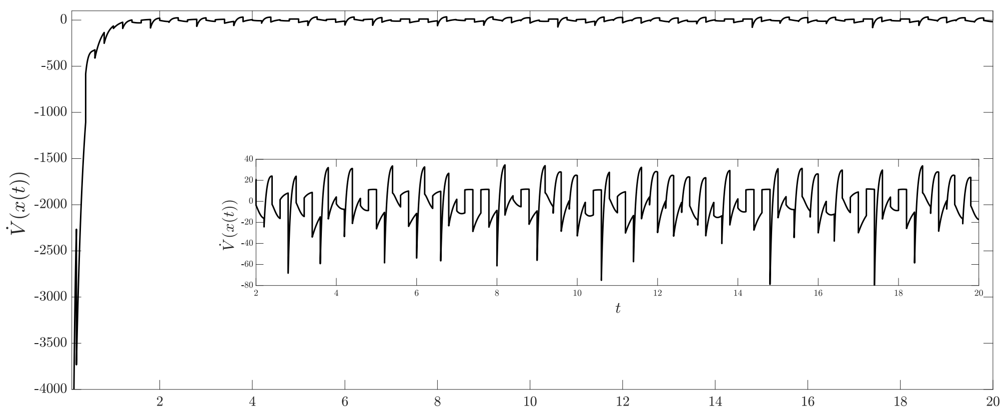

Figure 2: Function $\dot{V}$, defined by P_1 and d_1, along the switched affine system solution with initial condition $x_0 = [-20\ \ 20]'$.

to very conservative estimates of attractors. In order to overcome this difficulty, we will consider now the existence of multiple auxiliary scalar C^1 functions $V_p : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$V_p(x) = (x-d)'\,P_p\,(x-d)\,,\ \text{where } P_p \in \mathbb{R}^{n\times n} \text{ and } d \in \mathbb{R}^n. \tag{3.7}$$

Henceforth, the set of functions (3.7) will be called multiple auxiliary functions. In addition, we suppose that

$$\exists\ P_p > 0 \text{ such that } Q_p = A_p' P_p + P_p A_p < 0, \forall p \in \mathscr{P}. \tag{3.8}$$

Define $\mathscr{E}_p = \{x \in \mathbb{R}^n : \nabla V_p(x)(A_p x + b_p) \geq 0\}$ the set where the derivate of function V_p along the trajectories of subsystem p is positive or null. Let $\mathscr{E} = \bigcup_{p\in\mathscr{P}} \mathscr{E}_p$.

The next lemma provides sufficient conditions for the set $\mathscr{E}$ to be bounded.

Lemma 4. *Consider the switched affine system (3.1) and the multiple auxiliary functions V_p given by (3.7) such that (3.8) is satisfied. Then, the set $\mathscr{E}$ is bounded.*

Proof. The derivative of V_p along the solution of the subsystem $p \in \mathscr{P}$, is given by

$$\begin{aligned}\nabla V_p(x)(A_p x + b_p) &= x' Q_p x + 2(b_p' P_p - d' P_p A_p)x - 2d' P_p b_p \\ &\leq \lambda_{\max}(Q_p) x' x + 2\left\|b_p' P_p - d' P_p A_p\right\| \|x\| + 2\left|d' P_p b_p\right| \\ &= \lambda_{\max}(Q_p)\|x\|^2 + 2\kappa_p \|x\| + 2\zeta_p,\end{aligned}$$

where $\kappa_p = \left\|b_p' P_p - d' P_p A_p\right\|$ and $\zeta_p = \left|d' P_p b_p\right|$. Thus, we conclude that

$$\dot{V}_p(x) \leq \lambda_{\max}(Q_p)\|x\|^2 + 2\kappa_p \|x\| + 2\zeta_p. \tag{3.9}$$

Since (3.8) is satisfied for all $p \in \mathscr{P}$, we have that $\lambda_{\max}(Q_p) < 0$. Then, from (3.9), we conclude that the derivative of $V_p(x)$ is strictly negative when $\|x\| > -\frac{\kappa_p + \sqrt{\kappa_p^2 - 2\lambda_{\max}(Q_p)\zeta_p}}{\lambda_{\max}(Q_p)}$.

Then, $\mathscr{E}_p \subseteq \left\{x \in \mathbb{R}^n : 0 \leq \|x\| \leq -\frac{\kappa_p + \sqrt{\kappa_p^2 - 2\lambda_{\max}(Q_p)\zeta_p}}{\lambda_{\max}(Q_p)}\right\}$ and $\mathscr{E} = \bigcup_{p\in\mathscr{P}} \mathscr{E}_p \subseteq \{x \in \mathbb{R}^n : 0 \leq \|x\| \leq \eta\}$, where

$$\eta = \max_{p\in\mathscr{P}} \left\{ -\frac{\kappa_p + \sqrt{\kappa_p^2 - 2\lambda_{\max}(Q_p)\zeta_p}}{\lambda_{\max}(Q_p)} \right\}, \tag{3.10}$$

that is, the set $\mathscr{E}$ is bounded. □

The next lemma guarantees the existence of upper and lower bounds for the multiple auxiliary functions V_p given by (3.7).

Lemma 5. *Consider the switched affine system (3.1) and the multiple auxiliary functions V_p given by (3.7) such that (3.8) is satisfied. Then, there are continuous functions $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$ satisfying:*

$$\alpha(x) \leq V_p(x) \leq \beta(x), \quad \forall x \in \mathbb{R}^n \text{ and } \forall p \in \mathscr{P}. \tag{3.11}$$

Proof. To show the existence of functions α and β satisfying (3.11) we will determine a particular case of them. Since $P_p = P_p' > 0$, we have

$$\begin{aligned}V_p(x) &\leq \lambda_{\max}(P_p)(x-d)'(x-d) \\ &= (x-d)'\text{diag}[\lambda_{\max}(P_p), \cdots, \lambda_{\max}(P_p)](x-d),\end{aligned} \tag{3.12}$$

for all $p \in \mathscr{P}$ and $\forall x \in \mathbb{R}^n$. Define $\overline{P}_M = \text{diag}[\delta_{max}, \cdots, \delta_{max}]$, where $\delta_{max} = \max_{p\in\mathscr{P}}\{\lambda_{\max}(P_p)\}$. From (3.12), we have that

$$V_p(x) \leq (x-d)'\overline{P}_M(x-d), \quad \forall x \in \mathbb{R}^n \text{ and } \forall p \in \mathscr{P}. \tag{3.13}$$

Thus, considering $\beta(x) = (x-d)'\overline{P}_M(x-d)$, from (3.13) we have $V_p(x) \leq \beta(x)$, $\forall p \in \mathscr{P}$ and $\forall x \in \mathbb{R}^n$. Now, define $\overline{P}_m = \text{diag}[\delta_{min}, \cdots, \delta_{min}]$, where $\delta_{min} = \min_{p \in \mathscr{P}} \{\lambda_{\min}(P_p)\}$. Then, from (3.8), we have that

$$\begin{aligned} V_p(x) &\geq \lambda_{\min}(P_p)(x-d)'(x-d) \\ &= (x-d)'\text{diag}[\lambda_{\min}(P_p), \cdots, \lambda_{\min}(P_p)](x-d) \\ &\geq (x-d)'\overline{P}_m(x-d), \ \ \forall p \in \mathscr{P} \text{ and } \forall x \in \mathbb{R}^n. \end{aligned} \tag{3.14}$$

Define $\alpha(x) = (x-d)'\overline{P}_m(x-d)$. As a consequence of (3.14), we have $V_p(x) \geq \alpha(x)$, $\forall p \in \mathscr{P}$ e $\forall x \in \mathbb{R}^n$. Therefore, the scalar functions

$$\alpha(x) = (x-d)'\overline{P}_m(x-d) \text{ and } \beta(x) = (x-d)'\overline{P}_M(x-d),$$

satisfy (3.11). □

We now consider the continuous functions $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$, such that $\alpha(x) = (x-d)'P_m(x-d)$ and $\beta(x) = (x-d)'P_M(x-d)$, with $P_m, P_M \in \mathbb{R}^{n \times n}$ satisfying (3.11). Moreover, we define the sets $\Omega_{\ell_0}^{P_m,d} = \{x \in \mathbb{R}^n : \alpha(x) \leq \ell_0\}$, $\Omega_{\ell_j}^{P_m,d} = \{x \in \mathbb{R}^n : \alpha(x) \leq \ell_j\}$ and $\Theta^{P_M,d} = \{x \in \mathbb{R}^n : \beta(x) \leq \ell_0,\}$ with $\sup_{x \in \mathscr{E}} \beta(x) < \ell_0 < \infty$ and $\sup_{x \in \Omega_{\ell_{j-1}}^{P_m,d}} \beta(x) \leq \ell_j < \infty$, $j \in \{1, \cdots, \mathscr{N}+1\}$. It is clear by construction that

$$\mathscr{E} \subset \Theta^{P_M,d} \subseteq \Omega_{\ell_0}^{P_m,d} \subseteq \Omega_{\ell_1}^{P_m,d} \subseteq \cdots \subseteq \Omega_{\ell_j}^{P_m,d} \subseteq \Omega_{\ell_{j+1}}^{P_m,d} \subseteq \cdots \subseteq \Omega_{\ell_{\mathscr{N}+1}}^{P_m,d}. \tag{3.15}$$

The next lemma estimates the values $\ell_0, \cdots, \ell_{\mathscr{N}+1}$ and the regions $\mathscr{E}$, $\Theta^{P_M,d}$ and $\Omega_{\ell_j}^{P_m,d}$, $\forall j \in \{0, 1, \ldots, \mathscr{N}+1\}$.

Lemma 6. *Consider the switched affine system (3.1) and the multiple auxiliary functions V_p given by (3.7) such that (3.8) is satisfied. Moreover, assume that $\alpha(x) = (x-d)'P_m(x-d)$ and $\beta(x) = (x-d)'P_M(x-d)$, with $P_m, P_M \in \mathbb{R}^{n \times n}$ satisfying* (3.11)*, then:*

(i) If $\ell_0 > \lambda_{\max}(P_M)(\eta + \|d\|)^2$, then $\mathscr{E} \subset \Theta^{P_M,d} \subseteq \Omega_{\ell_0}^{P_m,d}$ where η is given by (3.10).

(ii) Given a real number ℓ_0 such that $\mathscr{E} \subset \Theta^{P_M,d} \subseteq \Omega_{\ell_0}^{P_m,d}$, then $\Omega_{\ell_{j-1}}^{P_m,d} \subseteq \Omega_{\ell_j}^{P_m,d}$, $\forall j \in \{1, \ldots, \mathscr{N}+1\}$, if $\ell_j \geq \frac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)} \ell_{j-1}$.

Proof.

(i) Due to Lemma 4, the inclusion $\mathscr{E} \subseteq \{x \in \mathbb{R}^n : 0 \leq \|x-d\| \leq \eta\}$, where η is given by (3.10), is verified. Then, when we analyze the values the continuous function $\beta : \mathbb{R}^n \to \mathbb{R}$, given by $\beta(x) = (x-d)'P_M(x-d)$, assumes in $\mathscr{E}$, we obtain

$$\beta(x) \leq \lambda_{\max}(P_M)\|x-d\|^2 \leq \lambda_{\max}(P_M)(\eta + \|d\|)^2, \forall x \in \mathscr{E}.$$

Thus, for $\ell_0 \in \mathbb{R}$ satisfying $\ell_0 > \lambda_{\max}(P_M)(\eta + \|d\|)^2$, we conclude that $\mathscr{E} \subset \Theta^{P_M,d}$. Therefore, by construction of the set $\Omega_{\ell_0}^{P_m,d}$, we have that $\mathscr{E} \subset \Theta^{P_M,d} \subseteq \Omega_{\ell_0}^{P_m,d}$.

(ii) The proof will be given by induction on the index $j \in \{1,\cdots,\mathcal{N}+1\}$. For $N=1$, we can show that $\Omega_{\ell_0}^{P_m,d} \subseteq \Omega_{\ell_1}^{P_m,d}$, when $\ell_1 \geq \frac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_0$. In fact, if $x \in \Omega_{\ell_0}^{P_m,d}$, then $\lambda_{\min}(P_m)||x-d||^2 \leq \alpha(x) \leq \ell_0, \forall x \in \Omega_{\ell_0}^{P_m,d}$. Hence, $\|x-d\|^2 \leq \left(\frac{\ell_0}{\lambda_{\min}(P_m)}\right)$, $\forall x \in \Omega_{\ell_0}^{P_m,d}$. For $x \in \Omega_{\ell_0}^{P_m,d}$, we have that

$$\beta(x) \leq \lambda_{\max}(P_M)\|x-d\|^2 \leq \lambda_{\max}(P_M)\left(\frac{\ell_0}{\lambda_{\min}(P_m)}\right) \leq \frac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_0,$$

for all $x \in \Omega_{\ell_0}^{P_m,d}$.

Thus, by defining $\ell_1 \geq \frac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_0$, we have that $\Theta^{P_M,d} \subseteq \Omega_{\ell_0}^{P_m,d} \subseteq \Omega_{\ell_1}^{P_m,d}$ since $\sup_{x\in\Omega_{\ell_0}^{P_m,d}} \beta(x) \leq \ell_1 < \infty$ is verified.

Next, we assume the result holds for $\mathcal{N}$ subsets, that is, the real numbers $\ell_0 \in \mathbb{R}$ and $\ell_j \geq \frac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_{j-1}$, $j \in \{1,\ldots,\mathcal{N}\}$, ensure that $\Omega_{\ell_0}^{P_m,d} \subseteq \Omega_{\ell_1}^{P_m,d} \subseteq \cdots \subseteq \Omega_{\ell_{\mathcal{N}-1}}^{P_m,d} \subseteq \Omega_{\ell_{\mathcal{N}}}^{P_m,d}$. Now, we show that the result holds for $\mathcal{N}+1$. For all $x \in \Omega_{\ell_{\mathcal{N}}}^{P_m,d}$, we have that $\|x-d\|^2 \leq \left(\frac{\ell_{\mathcal{N}}}{\lambda_{\min}(P_m)}\right)$. For $x \in \Omega_{\ell_{\mathcal{N}}}^{P_m,d}$, the following inequalities are satisfied $\beta(x) \leq \lambda_{\max}(P_M)\|x-d\|^2 \leq \left[\frac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\right]\ell_{\mathcal{N}}$, for all $x \in \Omega_{\ell_{\mathcal{N}}}^{P_m,d}$. Therefore, for $\ell_{\mathcal{N}+1} \geq \frac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_{\mathcal{N}}$, we have $\Omega_{\ell_{\mathcal{N}}}^{P_m,d} \subseteq \Omega_{\ell_{\mathcal{N}+1}}^{P_m,d}$. □

In order to take into account multiple auxiliary functions, we consider the following assumption.

Assumption 3.1. *For every pair of consecutive switching times $\tau_h < \tau_j$ such that $\sigma(\tau_h) = \sigma(\tau_j) = p$ the following holds:*

$$V_p(\varphi_p(\tau_h,x_0)) > V_p(\varphi_p(\tau_j,x_0)), \quad \text{if } \varphi_p(\tau_h,x_0) \notin \Theta^{P_M,d} \text{ and } \varphi_p(\tau_j,x_0) \notin \Theta^{P_M,d}.$$

The next result shows that every solution of the affine switched system (3.1) is bounded.

Lemma 7. *Consider the switched affine system (3.1) and the multiple auxiliary functions V_p given by (3.7) such that (3.8) is satisfied. Moreover, we assume that Assumption 3.1 is satisfied. Then, every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$, $x_0 \in \mathbb{R}^n$, is bounded.*

Proof. Let $\ell_0 \in \mathbb{R}$ such that $\ell_0 > \lambda_{\max}(P_M)(\eta + \|d\|)^2$ and η is given by (3.10). For $x_0 \in \Omega_{\ell_0}^{P_m,d}$, let $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ be a solution of the switched system (3.1) under arbitrary dwell-time switching signals. Then, by Lemma 3 in [17], we have that every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ stays inside $\Omega_{\ell_0}^{P_m,d}$, $\forall t \geq 0$, that is, the solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}$ is bounded.

Now, let $x_0 \notin \Omega_{\ell_0}^{P_m,d}$ and $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$. If $\varphi_{\sigma(t)}(t,x_0)$ enters $\Omega_{\ell_0}^{P_m,d}$ at some t, then the result follows from the first part of this proof. Suppose that the solution $\varphi_{\sigma(t)}(t,x_0) \notin \Omega_{\ell_0}^{P_m,d}$, $\forall t \geq 0$. Let $L_0 \in \mathbb{R}$ such that $\sup_{x\in\mathscr{E}} \beta(x) < \ell_0 < L_0$ and $x_0 \in \Omega_{L_0}^{P_m,d} = \{x \in \mathbb{R}^n : \alpha(x) \leq L_0\}$. Define $\Omega_{L_j}^{P_m,d} = \{x \in \mathbb{R}^n : \alpha(x) \leq L_j\}$ with $\sup_{x\in\Omega_{L_{j-1}}^{P_m,d}} \beta(x) \leq L_j < \infty$, $j \in \{1,\cdots,\mathcal{N}+1\}$. Then the following inclusion holds $\mathscr{E} \subset \Theta^{P_M,d} \subseteq \Omega^{P_m,d}_{L_0} \subseteq \Omega^{P_m,d}_{L_1} \subseteq \cdots \subseteq \Omega^{P_m,d}_{L_j} \subseteq \Omega^{P_m,d}_{L_{j+1}} \subseteq \cdots \subseteq \Omega^{P_m,d}_{L_{\mathscr{N}+1}}$. Due to the existence of the multiple functions V_p given by (3.7) and Assumption 3.1, the Lemma 3 in [17] again implies that if $x_0 \in \Omega^{P_m,d}_{L_0}$, then $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ stays inside $\Omega^{P_m,d}_{L_{\mathscr{N}+1}}$, $\forall t \geq 0$, that is, the solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ with $x_0 \notin \Omega^{P_m,d}_{\ell_0}$ is bounded.

Therefore, every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ of the switched system (3.1) under arbitrary dwell-time switching signal $\sigma(t)$ is bounded. □

Exploring the above results, next theorem establishes an extension of the invariance principle by means of multiple auxiliary scalar functions.

Theorem 8. *Consider the switched affine system (3.1) and the multiple auxiliary functions V_p given by (3.7) such that (3.8) is satisfied. Moreover, we assume that Assumption 3.1 is satisfied. Then every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$, $x_0 \in \mathbb{R}^n$, is attracted to the largest weakly invariant set of $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$.*

Proof. First, we consider $x_0 \in \Theta^{P_M,d}$. Note that, in the hypotheses of this theorem, the Assumption 3.1 and the inequalities (3.7) and (3.8) are satisfied. Then, by Lemma 7 and Lemma 3 in [17], we have that every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ is bounded and stays inside $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$ for all $t \geq 0$. By Proposition 2.1 we conclude that the solution will be attracted to a weakly invariant set in $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$.

Now let $x_0 \notin \Theta^{P_M,d}$ and $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$. If there exists $\bar{t} > 0$ such that $\varphi_{\sigma(\bar{t})}(\bar{t},x_0) \in \Theta^{P_M,d}$, then the proof follows from the first part of this proof. Suppose the solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ does not enter $\Theta^{P_M,d}$. Due to Lemma 7, we have that solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ is bounded. Consider the subsequence of switching times $\{\tau_{k_p}\}$ at which the system p becomes active, that is, $\sigma(\tau_{k_p}) = p$. From Assumption 3.1, we have that $V_p(\varphi_{\sigma(\tau_{k_p})}(\tau_{k_p},x_0))$ is a decreasing sequence of real numbers bounded from bellow. Then, $V_p(\varphi_{\sigma(\tau_{k_p})}(\tau_{k_p},x_0)) \to r_p$ where $k \to +\infty$ for all $p \in \mathscr{P}$. By Proposition 2.1, $\omega^+_\sigma(x_0)$ is a nonempty and weakly invariant set. Let $c \in \omega^+_\sigma(x_0)$, then there exists a sequence $\{t_j\}$ such that $\varphi_{\sigma(t_j)}(t_j,x_0) \to c$ as $j \to \infty$. Since the set $\mathscr{P}$ is finite, there exists at least one index $\overline{p} \in \mathscr{P}$ and a subsequence $\{t_{j_i}\}$ such that $t_{j_i} \in I_{\overline{p}}$. Then, $V_{\overline{p}}(\varphi_{\sigma(t_{j_i})}(t_{j_i},x_0)) \to V_{\overline{p}}(c) = r_{\overline{p}}$ for all $c \in \omega^+_\sigma(x_0)$. Using the same ideas of the proof of Proposition 2 in [3], we can guarantee the existence of an interval $[\varepsilon,\gamma]$ containing the origin and functions $\upsilon_j(t) = \varphi_{\sigma(t+t_j)}(t+t_j,x_0)$ defined on $[\varepsilon,\gamma]$, satisfying the following properties: $\upsilon_j(t)$ uniformly converges to $\upsilon(t)$ on $[\varepsilon,\gamma]$, $\upsilon(t) \subset \omega^+_\sigma(x_0)$ for all $t \in [\varepsilon,\gamma]$, $\dot{\upsilon}(t) = A_{\overline{p}}(\upsilon(t)) + b_{\overline{p}}$ and $\upsilon(0) = c$. Then $V_{\overline{p}}(\upsilon(t)) = r_{\overline{p}}$ and $\nabla V_{\overline{p}}(\upsilon(t))\,[A_{\overline{p}}(\upsilon(t)) + b_{\overline{p}}] = 0$ for all $t \in [\varepsilon,\gamma]$. Particularly, for $t = 0$, $\nabla V_{\overline{p}}(\upsilon(0))\,[A_{\overline{p}}(\upsilon(0)) + b_{\overline{p}}] = \nabla V_{\overline{p}}(c)\,[A_{\overline{p}}(\upsilon(c)) + b_{\overline{p}}] = 0$, then $c \in \{x \in \mathbb{R}^n : \nabla V_p(x)(A_p x + b_p) = 0\}$ and $\omega^+_\sigma(x_0) \subset \{x \in \mathbb{R}^n : \nabla V_p(x)(A_p x + b_p) = 0\} \subseteq \Theta^{P_M,d}$. The set $\omega^+_\sigma(x_0)$ is a weakly invariant set, then the solution is attracted to the largest weakly invariant in $\{x \in \mathbb{R}^n : \nabla V_p(x)(A_p x + b_p) = 0\}$, which leads to a contradiction because $\{x \in \mathbb{R}^n : \nabla V_p(x)(A_p x + b_p) = 0\} \subseteq \Theta^{P_M,d}$. Thus, there exists $\tilde{t} \in \mathbb{R}$ such that $\varphi(\tilde{t},x_0) \in \Theta^{P_M,d}$ and the result follows from the first part of this proof.

Therefore, every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ is attracted to the largest weakly invariant set in $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$. □

The following example illustrates Theorem 8.

Example 3.2. *([12]) Consider the affine switched system*

$$\dot{x} = A_{\sigma(t)}x + b_{\sigma(t)}, \qquad x \in \mathbb{R}^2, \tag{3.16}$$

where, $\sigma(t) \in \mathscr{P} = \{1,2\}$ and $A_1 = \begin{bmatrix} -4 & 1 \\ 2 & -7 \end{bmatrix}$, $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -7 & -5 \\ 3 & 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$. The eigenvalues of the matrices A_p, $p \in \{1,2\}$, are $\{-3.4384, -7.5616\}$ and $\{-3.5 \pm 1.6583i\}$, respectively. In addition, the equilibrium points of each subsystem p, $p \in \{1,2\}$, are given by: $x_{eq_1} = [0.5769\ \ 0.3077]'$ and $x_{eq_2} = [1\ \ -1.4]'$.

With the objective of obtaining an estimate of the attractor set for the switched affine system (3.16), *consider the auxiliary functions* (3.7), *with $P_1 = P_{11} = \begin{bmatrix} 0.6507 & 0.1375 \\ 0.1375 & 0.3493 \end{bmatrix}$ and $P_2 = P_{21} = \begin{bmatrix} 0.1133 & 0.0688 \\ 0.0688 & 0.3475 \end{bmatrix}$ satisfying* (3.8) *and the vector $d = d_1 = [1\ \ 0.5]'$. From Lemma 6, we can conclude that $\mathscr{E} \subset \Theta^{P_{M_1},d_1} \subseteq \Omega^{P_{m_1},d_1}_{\hat{\ell}_0} \subseteq \Omega^{P_{m_1},d_1}_{\hat{\ell}_1} \subseteq \Omega^{P_{m_1},d_1}_{\hat{\ell}_2} \subseteq \Omega^{P_{m_1},d_1}_{\hat{\ell}_3}$, as $\ell_0 = \hat{\ell}_0 = 9.3252 > \lambda_{\max}(P_{M_1})(\eta + \|d_1\|)^2$, where η is given by* (3.10), *$\hat{\ell}_1 = 22.1834$, $\hat{\ell}_2 = 52.7716$, $\hat{\ell}_3 = 125.5371$, $P_M = P_{M_1} = \begin{bmatrix} 0.7040 & 0 \\ 0 & 0.7040 \end{bmatrix}$ and $P_m = P_{m_1} = \begin{bmatrix} 0.2960 & 0 \\ 0 & 0.2960 \end{bmatrix}$. Then, from Theorem 8, every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ is attracted to a weakly invariant set in $\Omega^{P_{m_1},d_1}_{\hat{\ell}_3}$. Therefore, the attractor set of the system* (3.16) *is contained in the ellipsoidal region $\Omega^{P_{m_1},d_1}_{\hat{\ell}_3}$ for any dwell-time switching signal. The volume of this estimation is vol $\left(\Omega^{P_{m_1},d_1}_{\hat{\ell}_3}\right) = 1332.58$.*

Figure 3 illustrates $\Omega^{P_{m_1},d_1}_{\hat{\ell}_3}$ and a trajectory starting at $x_0 = [110\ \ 85]'$ with switching signal $\sigma(t)$ with dwell-time $h = 0.2$ seconds. This figure confirms the results of Theorem 8 by showing an attractor inside the set $\Omega^{P_{m_1},d_1}_{\hat{\ell}_3}$. Function $\nabla V_p(x)(A_px + b_p)$, $p \in \mathscr{P}$, along the solution of the switched affine system (3.16) *is shown in Figure 4. Observe in Figure 4 the changes of sign of the derivate of V along the solution.*

4 ESTIMATING THE ATTRACTOR SET BY OPTIMIZATION

In this section, the results of Section 3 are explored to obtain a systematic method to find the common auxiliary function or multiple auxiliary functions for the switched affine system in order to determine an estimate of the attractor as small as possible. For this purpose, a constrained optimization problem where the restrictions are given by the sufficient conditions of the invariance principle has been considered. Using this new procedure, Examples 3.1 and 3.2 are solved

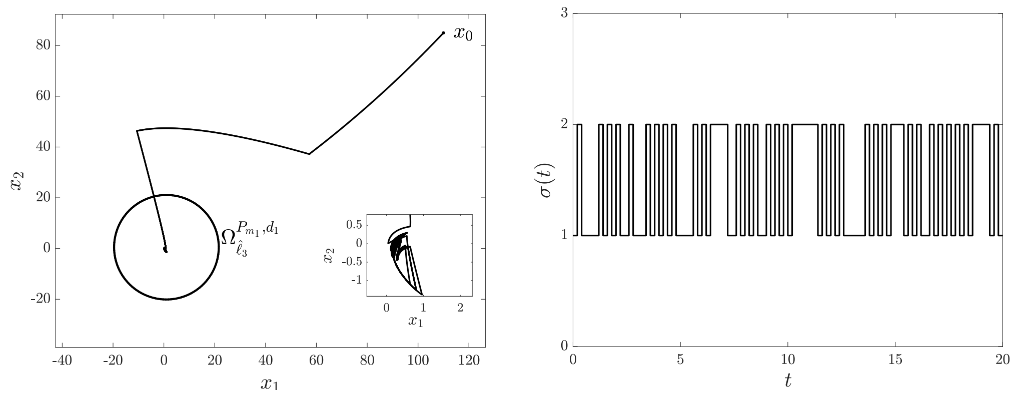

Figure 3: Phase portrait for Example 3.2 with initial condition $x_0 = [110 \ \ 85]'$ illustrating the level set $\Omega_{\hat{\ell}_3}^{P_{m_1},d_1}$ and switching signal with dwell-time $h = 0.2$ seconds.

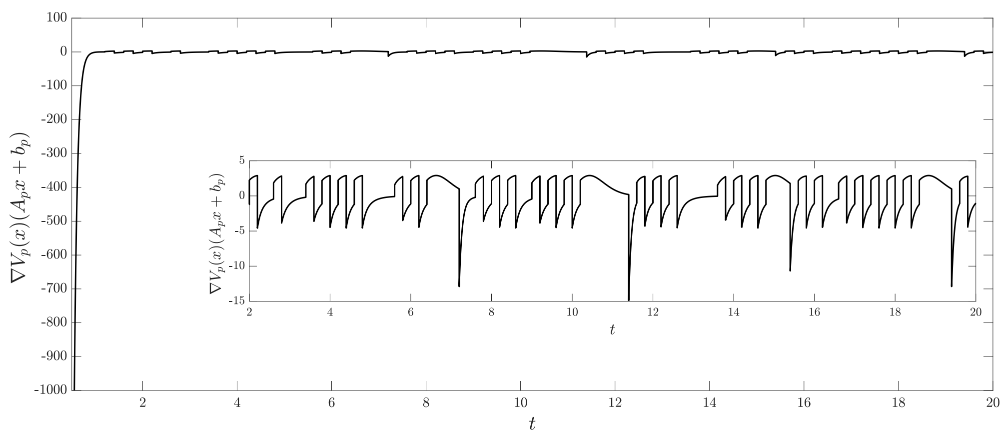

Figure 4: Function $\nabla V_p(x)(A_p x + b_p)$, defined by P_{11}, P_{21} and d_1, along the switched affine system solution with initial condition $x_0 = [110 \ \ 85]'$.

again to show that the new estimates of the attractor have smaller volume than the estimates obtained previously by trial and error. To obtain the solution of the optimization problems in the next examples, we have used the function *ga*, of the Global Optimization Toolbox of Matlab, which is a Genetic Algorithm which explores the technique of heuristic optimization, inspired by biological evolution, to solve the optimization problem [16].

4.1 Common auxiliary function

Theorem 3 ensures that the sublevel set $\Omega_{\ell}^{P,d}$, associated with the common auxiliary function (3.2), is an estimate of the attractor set for the switched affine system (3.1) under arbitrary dwell-time switching. However, it is clear from Lemma 1 and Example 3.1 that the size of set $\Omega_\ell^{P,d}$ is related to matrix $P \in \mathbb{R}^{n\times n}$ and the vector $d \in \mathbb{R}^n$.

Thus, we are interested in finding a matrix $P > 0$ and a vector $d \in \mathbb{R}^n$ such as (P,d) minimizes the volume of the set $\Omega_\ell^{P,d}$. For this purpose, we considered the next optimization problem. Note that this problem can be constructed due to the format of the assumptions of Theorem 3.

Optimization Problem 4.1.

$$\textit{minimize} \qquad -\ln(\det(P)) \tag{4.1}$$

$$\textit{subject to} \qquad P > 0 \tag{4.2}$$

$$Q_p < 0, \forall p \in \mathscr{P} \tag{4.3}$$

$$\lambda_{\max}(P_M)(z_p + \|d\|)^2 - 1 < 0, \forall p \in \mathscr{P} \tag{4.4}$$

where

$$P \in \mathbb{R}^{n\times n}, d \in \mathbb{R}^n,\ Q_p = A_p'P + PA_p \in \mathbb{R}^{n\times n}, \forall p \in \mathscr{P},$$
$$\mu_p = \left\|b_p'P - d'PA_p\right\|,\ \forall p \in \mathscr{P},\ \xi_p = \left|d'Pb_p\right|,\ \forall p \in \mathscr{P},$$
$$z_p = -\frac{\mu_p + \sqrt{\mu_p^2 - 2\lambda_{\max}(Q_p)\xi_p}}{\lambda_{\max}(Q_p)}, \forall p \in \mathscr{P}.$$

The next theorem establishes the formulation for finding an estimate of the attractor set of the switched affine system (3.1) with minimum volume. In this theorem, the estimate of the attractor set is formulated into an optimization problem.

Theorem 1. *Suppose that the pair (P,d) is a solution for the Optimization Problem 4.1. Then, $\Omega_1^{P,d}$ is an estimate of the attractor set for the switched affine system (3.1) with minimum volume, that is, every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ is attracted to a weakly invariant set in $\Omega_1^{P,d}$.*

Proof. Let $P \in \mathbb{R}^{n\times n}$ and $d \in \mathbb{R}^n$ be a solution to the Optimization Problem 4.1. Then,

$$\Omega_1^{P,d} = \left\{x \in \mathbb{R}^n : (x-d)'P(x-d) < 1\right\} = \left\{x \in \mathbb{R}^n : (x-d)'\overline{P}(x-d) < \ell\right\} = \Omega_\ell^{\overline{P},d},$$

where $P = \frac{1}{\ell}\overline{P}$. Moreover, the constraints of the optimization problem (4.2)−(4.3) are equivalent to (3.3) and (3.4). Thus, from Theorem 3, it follows that every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ is attracted to a weakly invariant set in $\Omega_1^{P,d}$. Since the volume of $\Omega_1^{P,d}$ is proportional to $(\det(P))^{1/2}$ [4], minimizing this determinant is equivalent to minimizing $-\ln(\det(P))$ and therefore, the proof is complete. □

We can obtain the matrix P and the vector d satisfying Theorem 1 by solving the Optimization Problem (4.1) via numerical algorithms. In other words, a computational procedure based on nonlinear optimization to estimate the attractor set for the switched affine systems, under arbitrary dwell-time switching, is obtained by exploring Theorem 1.

The next Procedure 4.1 explores Theorem 1 to estimate the attractor set of the switched affine systems (3.1) under arbitrary dwell-time switching signals.

Procedure 4.1.

- ***Input:*** $A_p \in \mathbb{R}^{n\times n}$, $b_p \in \mathbb{R}^n$, $p \in \mathscr{P}$.
- ***Output:*** $\Omega_1^{P,d}$ *(an estimate the attractor set of switched affine systems (3.1) obtained via Theorem 1).*

  1. *Find the positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *and the vector* $d \in \mathbb{R}^n$*, solving Optimization Problem 4.1.*
  2. *Calculate the volume of the set* $\Omega_1^{P,d}$*.*

Example 4.1 explores Theorem 4.1 and Procedure 4.1 to solve Example 3.1 again, in order to obtain a better estimate of the attractor set of the switched affine system (3.6) under arbitrary switching signal.

Example 4.1. *Consider the switched affine system (3.6) presented in Example 3.1. Following the Procedure 4.1, we can find the local optimal solution* $P = \begin{bmatrix} 0.0246 & -0.0006 \\ -0.0006 & 0.0394 \end{bmatrix}$ *and* $d = \begin{bmatrix} 0.0841 \\ 0.7245 \end{bmatrix}$*, which defines the ellipsoidal region* $\Omega_1^{P,d}$ *centered at d with* $vol\left(\Omega_1^{P,d}\right) = 100.8775$*.*

Then, from Theorem 4.1, every solution $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ *is attracted to a weakly invariant set in* $\Omega_1^{P,d}$*. Therefore, the attractor set of the system* (3.6) *is contained in the ellipsoidal region* $\Omega_1^{P,d}$ *for any dwell-time switching signal. Moreover, we can confirm that the estimate of the attractor set obtained by using Procedure 4.1 is better than those presented in Example 3.1, whose volume is* $vol\left(\Omega_{\bar{\ell}}^{P_1,d_1}\right) = 680.7098$*.*

Figure 5 illustrates the trajectory of the switched affine system with $x_0 = [-15 \;\; 27]'$ *under a dwell-time switching signal with* $h = 0.2$ *seconds, and, the estimate of* $\Omega_{\bar{\ell}}^{P_1,d_1}$*, obtained in Example 3.1, and* $\Omega_1^{P,d}$*, obtained by using Procedure 4.1. The attractor set is contained in* $\Omega_1^{P,d}$*, confirming the results of Theorem 4.1.*

4.2 Multiple auxiliary functions

The results established in Subsection 8 ensure that the set $\Omega_{\ell_{\mathscr{N}+1}}^{P_m,d}$, associated with the scalar function $\alpha(x)$, given by $\alpha(x) = (x-d)P_m(x-d)$, where $P_m \in \mathbb{R}^{n\times n}$, is an estimate of the attractor set of the switched system affine (3.1) for any dwell-time switching signal $\sigma(t)$. However, it is evident from the hypotheses of Theorem 8 that the size of $\Omega_{\ell_{\mathscr{N}+1}}^{P_m,d}$, is related to the positive definite matrices $P_1, \cdots, P_{\mathscr{N}}$, P_m, $P_M \in \mathbb{R}^{n\times n}$ and the vector $d \in \mathbb{R}^n$. Then, at this moment, we are interested in finding matrices $P_1, \cdots, P_{\mathscr{N}}$, P_m, $P_M \in \mathbb{R}^{n\times n}$ and vector $d \in \mathbb{R}^n$ that minimize the

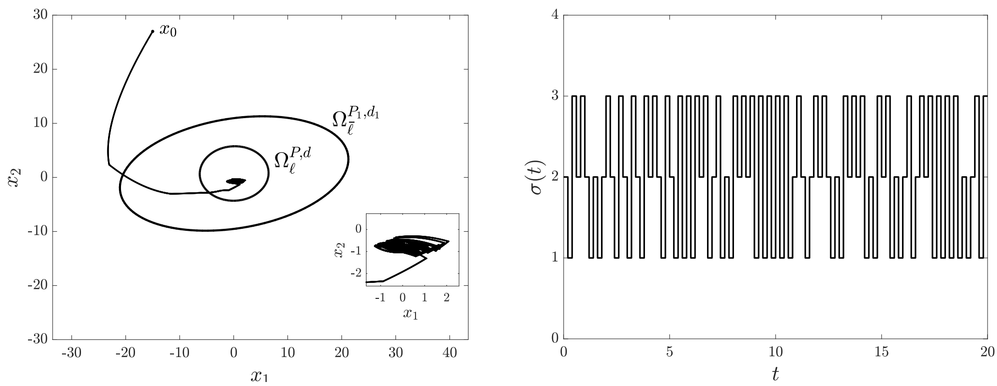

Figure 5: The estimates $\Omega_1^{P,d}$ and $\Omega_{\bar{\ell}}^{P_1,d_1}$ of the attractor set and a solution $\varphi_{\sigma(t)}(t,x_0)$, $x_0 = [-15\ \ 27]'$, of the switched affine system (3.6) under a switching signal $\sigma(t)$ with dwell-time $h = 0.2$ seconds.

volume of the set $\Omega_{\ell_{\mathscr{N}+1}}^{P_m,d}$. For this purpose, we consider the next optimization problem. Note that this problem can be constructed due to the format of the assumptions of Theorem 8.

Optimization Problem 4.2.

minimize $\quad -\ln(\det(P_m)) \quad (4.5)$

subject to

$$P_p > 0,\ \forall p \in \mathscr{P} \tag{4.6}$$

$$Q_p < 0,\ \forall p \in \mathscr{P} \tag{4.7}$$

$$P_m - P_p < 0,\ \forall p \in \mathscr{P} \tag{4.8}$$

$$P_p - P_M < 0,\ \forall p \in \mathscr{P} \tag{4.9}$$

$$\lambda_{\max}(P_M)\,(\eta_p + \|d\|)^2 - 1 < 0,\ \forall p \in \mathscr{P} \tag{4.10}$$

$$P_m > 0 \tag{4.11}$$

$$P_M > 0 \tag{4.12}$$

where

$$P_p \in \mathbb{R}^{n\times n}, \forall p \in \mathscr{P}, P_m, \in \mathbb{R}^{n\times n}, P_M \in \mathbb{R}^{n\times n}, d \in \mathbb{R}^n,$$

$$Q_p = A_p' P_p + P_p A_p, \forall p \in \mathscr{P},$$

$$\kappa_p = \left\| b_p' P_p - d' P_p A_p \right\|, \forall p \in \mathscr{P},$$

$$\zeta_p = \left| d' P_p b_p \right|, \forall p \in \mathscr{P},$$

$$\eta_p = -\frac{\kappa_p + \sqrt{\kappa_p^2 - 2\lambda_{\max}(Q_p)\zeta_p}}{\lambda_{\max}(Q_p)}, \forall p \in \mathscr{P}.$$

The next result allows us to find an estimate of the attractor set of switched affine system 3.1 with minimum volume under arbitrary switching signal by exploring Theorem 8. In this result, the estimation of the attractor set is formulated as a nonlinear optimization problem.

Theorem 2. *Suppose that* $(P_1,\ldots,P_{\mathscr{N}},P_m,P_M,d)$ *is a solution of the Optimization Problem 4.2. Consider that Assumption 3.1 is satisfied and* $\ell_j \geq \dfrac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_{j-1}$, $\forall j \in \{1,\ldots,\mathscr{N}+1\}$. *Then,* $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$ *is an estimate of the attractor set of the switched affine system (3.1) with minimum volume for any arbitrary dwell-time switching signal, that is, every solution* $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$ *is attracted to a weakly invariant set in* $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$.

Proof. Consider that $(P_1,\ldots,P_{\mathscr{N}},P_m,P_M,d)$ is a solution for the Optimization Problem 4.2. From constraints (4.6) - (4.7) and from the first $\mathscr{N}$ coordinates of the solution of the Optimization Problem 4.2, it is possible to write the functions V_p as (3.7) such that (3.8) is satisfied. Using (4.8) and (4.9), we can define $\alpha(x) = (x-d)'P_m(x-d)$ and $\beta(x) = (x-d)'P_M(x-d)$ satisfying (3.11). Rewriting (4.10), we have $\lambda_{\max}(P_M)(\eta_p + \|d\|)^2 < 1$, $\forall p \in \mathscr{P}$, that is, by Lemma 6, one guarantees that (3.15) is satisfied, where $\ell_0 = 1$, $\ell_j \geq \dfrac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_{j-1}$, $\forall j \in \{1,\ldots,\mathscr{N}+1\}$ and η is given by (3.10). Since Assumption 3.1 is considered, every hypothesis of Theorem 2 is satisfied. Therefore, every solution of the switched affine system (3.1) under arbitrary *dwell-time* switching signal, $\varphi_{\sigma(t)}(t,x_0) \in \mathscr{S}_{dwell}$, with $x_0 \in \mathbb{R}^n$, is attracted to the largest invariant set in $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$. Since the volume of $\Omega^{P_m,d}_1$ is proportional to $(\det(P_m))^{1/2}$ [4], minimizing this determinant is equivalent to minimizing $-\ln(\det(P_m))$ and the proof is complete. □

Positive definite matrices $P_1,\ldots,P_{\mathscr{N}}$, P_m, $P_M \in \mathbb{R}^{n\times n}$ and a vector $d \in \mathbb{R}^n$, which satisfy Theorem 2, are obtained by numerically solving the Optimization Problem 4.2. In other words, $(P_1,\ldots,P_{\mathscr{N}},P_m,P_M,d)$ can be systematically calculated to obtain a good estimate of the attractor set.

Exploring Theorem 2, the next procedure is defined to estimate the attractor set of switched affine systems (3.1) under arbitrary dwell-time switching.

Procedure 4.2.

- ***Input:*** $A_p \in \mathbb{R}^{n\times n}$, $b_p \in \mathbb{R}^n$, $p \in \mathscr{P}$.
- ***Output:*** $\Omega^{P_m,d}_{\ell_{\mathscr{N}+1}}$ *(estimate of the attractor set of the system (3.1) obtained via Theorem 2.)*

  1. *Find the positive definite matrices* P_1, $\ldots$, $P_{\ell_{\mathscr{N}}}$, P_m, $P_M \in \mathbb{R}^{n\times n}$ *and the vector* $d \in \mathbb{R}^n$, *by solving the Optimization Problem 4.2.*
  2. *Since* $\ell_0 = 1$, *for* $j \in \{1,\ldots,\mathscr{N}+1\}$,

     $*$ *calculate* $\ell_j \geq \dfrac{\lambda_{\max}(P_M)}{\lambda_{\min}(P_m)}\ell_{j-1}$.

3. Calculate the volume of the set $\Omega_{\ell_{\mathcal{N}+1}}^{P_m,d}$.

Example 4.2 explores Theorem 2 under Procedure 4.2 to obtain a better estimate of the attractor set as compared to the estimate obtained in Example 3.2.

Example 4.2. *Consider the switched affine system of* (3.16) *presented in Example 3.2. Using the Procedure 4.2 and solving the Optimization Problem 4.2, we obtain the local optimal solution* $P_1 = \begin{bmatrix} 0.4759 & 0.0938 \\ 0.0938 & 0.4983 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.5625 & 0.0170 \\ 0.0170 & 0.5663 \end{bmatrix}$, $P_m = \begin{bmatrix} 0.4758 & 0.0938 \\ 0.0938 & 0.4982 \end{bmatrix}$, $P_M = \begin{bmatrix} 0.5816 & 0 \\ 0 & 0.58160 \end{bmatrix}$, $d = \begin{bmatrix} -0.3110 \\ 0.1157 \end{bmatrix}$ *and the scalars* $\ell_0 = 1$, $\ell_1 = 1.4815$, $\ell_2 = 2.1951$ *and* $\ell_3 = 3.2522$. *Then, from Theorem 2, every solution* $\varphi_{\sigma(t)}(t, x_0) \in \mathscr{S}_{dwell}$ *is attracted to a weakly invariant set in* $\Omega_{\ell_3}^{P_m,d}$. *Therefore, the attractor set of the system* (3.16) *is contained in the ellipsoidal region* $\Omega_{\ell_3}^{P_m,d}$ *for any dwell-time switching signal. The volume of this estimation is* $vol\left(\Omega_{\ell_3}^{P_m,d}\right) = 21.3861$. *Moreover, we can confirm that the estimate of the attractor set obtained by using Procedure 4.2 is better than the one presented in Example 3.2, whose volume is* $vol\left(\Omega_{\hat{\ell}_3}^{P_{m_1},d_1}\right) = 1332.58$.

Figure 6 illustrates the trajectory of the switched affine system with $x_0 = [90 \;\; 27]'$ *under a dwell-time switching signal with* $h = 0.2$ *seconds, and, the estimate of* $\Omega_{\hat{\ell}_3}^{P_{m_1},d_1}$, *obtained in Example 3.2, and* $\Omega_{\ell_3}^{P_m,d}$, *obtained by using Procedure 4.2. The attractor set is contained in* $\Omega_{\ell_3}^{P_m,d}$, *confirming the results of Theorem 2.*

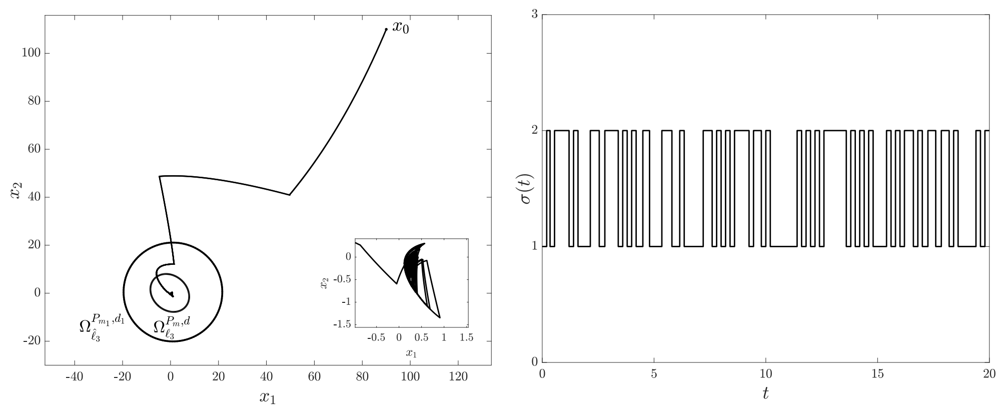

Figure 6: Phase portrait for Example 3.2 with initial condition $x_0 = [90 \;\; 27]'$ illustrating the level sets $\Omega_{\ell_3}^{P_m,d}$, $\Omega_{\hat{\ell}_3}^{P_{m_1},d_1}$ and switching signal with dwell-time $h = 0.2$ seconds.

5 CONCLUSION

In this paper, we have studied the asymptotic behavior of the solutions of the class of switched affine systems under arbitrary dwell-time switching signal exploring the specific structure of these systems.

The invariance principles proposed in this paper were obtained via a common auxiliary scalar function and multiple auxiliary scalar functions. These principles offer estimates of the attractor set of the switched affine systems (3.1) in terms of an ellipsoidal sublevel set for any dwell-time switching signal. Exploring the invariance principle and a nonlinear optimization problem, optimal estimates of the attractor set were obtained. Illustrative examples show the potential of the theoretical results in providing information on the asymptotic behavior of solutions of switched affine systems under arbitrary dwell-time switching signals.

ACKNOWLEDGEMENTS

This work was partially supported by the project INCT (National Institute of Science and Technology) under the grant FAPESP (São Paulo Research Foundation) 2014/5081-0 and by CNPq (National Council for Scientific and Technological Development) under the grant 308067/2017-7.

RESUMO. Neste artigo, uma abordagem para investigar o sistema chaveado afim por meio de desigualdades matriciais é apresentado. Particularmente, uma extensão do princípio de invariância de LaSalle para esta classe de sistemas sob sinal chaveamento dwell-time arbitrário é apresentado. Os resultados propostos empregam uma função escalar auxiliar comum e também múltiplas funções escalares auxiliares para estudar o comportamento assintótico das soluções chaveadas e estimar seus atratores para qualquer sinal de chaveamento dwell-time. Uma característica especifica destes resultados é que a derivada das funções escalares auxiliares podem assumir valores positivos em alguns conjuntos limitados. Além disso, um problema de otimização restrita é formulado para determinar numericamente as funções escalares auxiliares e minimizar o volume do atrator estimado. Exemplos numéricos mostram o potencial dos resultados teóricos em fornecer informações sobre o comportamento assintótico das soluções do sistema chaveado afim sob sinais de chaveamento dwell-time arbitrários.

Palavras-chave: sistema chaveado afim, princípio de invariância, dwell-time, conjunto de atrator.

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