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Energy Decay for the Solutions of a Coupled Wave System

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Abstract. In this work we establish existence, uniqueness and exponential decay of energy for the solutions of a system of wave equations coupled with locally distributed damping in a bounded smooth domain of any space dimension.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with smooth boundary $\partial\Omega$. Let $H^1(\Omega)$ and $L^2(\Omega) = H^0(\Omega)$ be the usual Sobolev spaces of Lebesgue square integrable functions defined in Ω with their usual norms (see[3]). Let $H_0^1(\Omega)$ be the subspace of $H^1(\Omega)$ of the functions vanishing on $\partial\Omega$. We consider the following system of coupled wave equations

$$
\begin{aligned}
u_{tt} - \Delta u + \alpha(x)(u_t - v_t) &= 0 \text{ in } \Omega \times [0,\infty), && (1.1)\\
v_{tt} - \Delta v + \alpha(x)(v_t - u_t) &= 0 \text{ in } \Omega \times [0,\infty), && (1.2)\\
u = v &= 0 \text{ in } \partial\Omega \times [0,\infty), && (1.3)\\
u(x,0) = u_0(x),\ u_t(x,0) &= u_1(x) \text{ in } \Omega, && (1.4)\\
v(x,0) = v_0(x),\ v_t(x,0) &= v_1(x) \text{ in } \Omega, && (1.5)
\end{aligned}
$$

where $\alpha(x)$ is a given function such that $\alpha \in W^{1,\infty}$, $\alpha(x) \geq 0$ *in* Ω and

$$
\alpha_0 = \int_\Omega \alpha(x)\, dx > 0. \qquad (1.6)
$$

This means that $\alpha(x)$ can vanishes at some points of Ω but the measure of its support is positive. Here Δ is the laplacian in the space variable x and sub-index t denotes partial derivative with respect to t.

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We prove that the system (1.1)-(1.6) has a unique solution in the class

$$(u,v) \in C^0([0,\infty), H_0^1(\Omega))^2 \cap C^1([0,\infty), L^2(\Omega))^2.$$

For such solutions the total energy is defined by

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + |v_t|^2 + |\nabla u|^2 + |\nabla v|^2 dx.$$

Our main goal is to prove that there exist positive constants C and w such that

$$E(t) \le C\, E(0) e^{-\,w\,t}, \text{ for every } t > 0.$$

We obtain such exponential decay estimate by using tools of the semigroup theory. A wave system with locally distributed damping coupled in parallel have also been been considered in [8] for the space domain $\Omega = (0,1) \subset \mathbb{R}$ with friction coefficient α constant. Our result extend the one in [8] in the sense that we allow higher dimension spaces and variable friction coefficient.

Stability for the one dimensional damped wave equation

$$u_{tt} - u_{xx} + \alpha(x)\, u_t = 0, \quad x \in (0,L), \quad \forall\, t > 0,$$

has been studied by many authors. See, for instance, [4] and the reference cited there. For similar results in higher dimension space we mention [5].

Stability for coupled wave system has been considered in [6], [1], [2] and [8] among others. In [6], both wave equations are damped on the boundary and the coupling is effected by compact operator. Exponential stability is obtained in [6] when the boundary damping is linear. Boundary damping is also considered in [1] and [2].

The rest of the paper is organized as follows. In the section 2 we prove existence and uniqueness of solution in the class mentioned. The section 3 is devoted to prove the exponential decay of the solutions.

2. Existence and Uniqueness of Solution

In order to make use of the Theory of Semigroups we write the system (1.1)-(1.5) in the abstract form,

$$\begin{aligned} \mathbf{U}_t - A\mathbf{U} &= 0, \\ \mathbf{U}(0) &= \mathbf{U}_0, \end{aligned}$$

where $\mathbf{U} = (u,\phi,v,\psi)^T$, $\phi = u_t$, $\psi = v_t$ and $A : [\,H_0^1 \times L^2\,]^2 \to [\,H_0^1 \times L^2\,]^2$ with $D(A) = [\,(H_0^1 \cap H^2) \times H_0^1\,]^2$, and

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ \Delta & -\alpha(x)\, I & 0 & \alpha(x)\, I \\ 0 & 0 & 0 & I \\ 0 & \alpha(x)\, I & \Delta & -\alpha(x) I \end{bmatrix}.$$

In the space $[\,H_0^1 \times L^2\,]^2$ we consider the canonical inner product, i.e., for

$$\mathbf{U}_j = (u_j, \phi_j, v_j, \psi_j), \quad j = 1, 2$$

we have

$$\langle \mathbf{U}_1, \mathbf{U}_2 \rangle = \int_\Omega \nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2 + \phi_1 \phi_2 + \psi_1 \psi_2 \, dx.$$

The operator A has an important property; it is dissipative as state the following:

Proposition 2.1. *The operator A is dissipative, i.e.,*

$$\langle A\mathbf{U}, \mathbf{U} \rangle \leq 0,$$

for every $\mathbf{U} \in [\,H_0^1 \times L^2\,]^2$.

Proof. Taking the inner product of $A\mathbf{U}$ and $\mathbf{U}$ we obtain,

$$\langle A\mathbf{U}, \mathbf{U} \rangle = -\int_\Omega \alpha(x) |u_t - v_t|^2 dx, \tag{2.7}$$

showing that A is dissipative. □

For the sake of completeness we state the well known Lumer-Phillips theorem, whose proof can be seen in [9].

Theorem 2.1. *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is a dissipative operator and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $(\lambda_0 I - A)$ is H, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.*

The following corollary of the Lummer-Phillips Theorem will be used soon after.

Corollary 2.1. *Let A be a linear operator with dense domain $D(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A) = \{\lambda \in \mathbb{C} : \text{ exist } (\lambda I - A)^{-1}\}$, the resolvent set of A, then A is the infinitesimal generator of a a C_0-semigroup of contractions on H.*

Proof. By the assumption $0 \in \rho(A)$, A is invertible and A^{-1} is compact in H. By the contraction mapping theorem, it is easy to see that the operator $\lambda I - A = A(\lambda A^{-1} - I)$ is invertible for $0 < \lambda < ||A^{-1}||$. Therefore, it follows from the Lumer-Phillips Theorem that A is the infinitesimal generator of a a C_0-semigroup of contractions on H. □

Now we can prove the following theorem.

Theorem 2.2. *The linear operator A generates a C_0-semigroup of contractions in $[\,H_0^1 \times L^2\,]^2$.*

Proof. The operator is densely defined and dissipative. Hence, we just need to prove $0 \in \rho(A)$. In order to do so, take

$$\mathbf{U} = (u, \phi, v, \psi)^T \in [\, H_0^1 \times L^2 \,]^2 \text{ and } F = (f_1, f_2, f_3, f_4)^T \in [\, H_0^1 \times L^2 \,]^2.$$

and consider the equation $A\mathbf{U} = F$, i.e.,

$$\left[\begin{array}{c} \phi \\ \Delta u - \alpha(x)(u_t - v_t) \\ \psi \\ \Delta v - \alpha(x)(v_t - u_t) \end{array} \right] = \left[\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4) \end{array} \right].$$

Then, we have

$$\phi = f_1 \in H_0^1, \tag{2.8}$$
$$\Delta u - \alpha(x)(u_t - v_t) = f_2 \in L^2, \tag{2.9}$$
$$\psi = f_3 \in H_0^1, \tag{2.10}$$
$$\Delta v - \alpha(x)(v_t - u_t) = f_4 \in L^2. \tag{2.11}$$

Using (2.8), (2.9) and (2.10) we obtain

$$\Delta u = \alpha(x)(f_1 - f_3) \in H_0^1. \tag{2.12}$$

From the standard Theory of Linear Elliptic Equations it follows that (2.12) has a unique solution in $H_0^1 \cap H^2$.

Now, using (2.8), (2.10) and (2.11) we get the equation

$$\Delta v = \alpha(x)(f_3 - f_1) \in H_0^1, \tag{2.13}$$

and by the same argument we see that (2.13) has a unique solution in $H_0^1 \cap H^2$.

Hence $\mathbf{U} \in D(A)$ and as a consequence $0 \in \rho(A)$. This ends the proof. □

From the Theory of Semigroups, it follows that the problem (1.1)-(1.6) has a unique solution with the following regularity;

$$(u, \phi, v, \psi) \in C^0([0, \infty) : [\, (H_0^1 \cap H^2) \times H_0^1 \,]^2) \cap C^1([0, \infty) : [\, H_0^1 \cap L^2 \,]^2).$$

Now, using some theorems of immersion from the classical theory of Sobolev Spaces [3] it follows that the solution satisfy;

$$(u, v) \in C^0([0, \infty) : H_0^1(\Omega_0)^2) \cap C^1([0, \infty) : L^2(\Omega_0)^2)),$$

as we stated in the introduction.

3. Exponential Decay

We study now, the decay for the solutions (u, v) of the system (1.1)-(1.6). In this sense consider the following theorem

Theorem 3.3. *Let $S(t)$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\{i\beta : \beta \in \mathbb{R}\} \subset \rho(A), \tag{3.1}$$

and

$$||(\lambda I - A)^{-1}|| \leq C \quad \forall \quad Re(\lambda) \geq 0. \tag{3.2}$$

Proof. See [10]. □

The main result of this note is the theorem 3.4 below. For the proof of this theorem we will use the technique developed in [7] to prove the exponential decay of the solution, in this sense, we will show the exponentially stability of the C_0-semigroup of contractions associated with the system (1.1)-(1.6).

Theorem 3.4. *Let (u, v) be the solution of (1.1)-(1.6) with initial data in $(H_0^1(\Omega) \times L^2((\Omega)))^2$. There exist positive constants C and w, independent of the initial data, such that*

$$E(t) \leq C\, E(0) e^{-w\,t}, \ \ \textit{for every } \ t > 0.$$

Proof. By theorem 3.3 it suffices to verify (3.1) and (3.2). To verify (3.1) we use the contradiction argument. In this sense, if (3.1) is not true, then there must be a $\beta \in \mathbb{R}$, such that $\beta \neq 0$ and $i\beta$ is in the spectrum of A. Since A^{-1} is compact, $i\beta$ must be an eigenvalue of A. It turns out there is a vector function

$$\mathbf{U} = (u, \phi, v, \psi)^T \in D(A), \quad ||\mathbf{U}|| \neq 0,$$

such that

$$i\beta\mathbf{U} - A\mathbf{U} = 0. \tag{3.3}$$

Taking the inner product of (3.3) with $\mathbf{U}$ in $(H_0^1 \times L^2)^2$ and taking its real part yields

$$Re(i\beta\mathbf{U} - A\mathbf{U}) = \int_\Omega \alpha(x)|\phi - \psi|^2 dx = 0.$$

From the following estimate

$$||\alpha\,(\phi - \psi)||^2 = \int_\Omega \alpha^2(x)|\phi - \psi|^2 dx \leq ||\alpha||_{L^\infty} \int_\Omega \alpha(x)|\phi - \psi|^2 dx = 0,$$

we have that

$$\alpha(x)(\phi - \psi) = 0 \text{ for all } x \in \Omega,$$

and then $\phi = \psi$ for all $x \in \Omega$. A contradiction.

For the proof of (3.2) notice that

$$||(\lambda I - A)^{-1}|| \leq C \Leftrightarrow ||(\lambda I - A)^{-1}\mathbf{F}|| \leq ||\mathbf{F}|| \text{ for all } \mathbf{F} \in (H_0^1 \times L^2)^2. \tag{3.4}$$

To prove (3.4) we used (3.1) and then there must be a $\lambda \in \rho(A)$ such that

$$\lambda\mathbf{U} - A\mathbf{U} = \mathbf{F} \text{ for all } \mathbf{F} \in (H_0^1 \times L^2)^2. \tag{3.5}$$

Denoting $\mathbf{F} = (f_1, f_2, f_3, f_4)^T$, from (3.5) we obtain

$$\begin{aligned} Re(\lambda)u - \phi &= f_1, & (3.6)\\ Re(\lambda)\phi - \Delta u + \alpha(x)(\phi - \psi) &= f_2, & (3.7)\\ Re(\lambda)v - \psi &= f_3, & (3.8)\\ Re(\lambda)\psi - \Delta v + \alpha(x)(\psi - \phi) &= f_4. & (3.9) \end{aligned}$$

multiplying (3.6), (3.7), (3.8), (3.9) respectively for $-\Delta u$, ϕ, $-\Delta v$, ψ and adding, we obtain

$$-Re(\lambda)u\Delta u - Re(\lambda)v\Delta v + Re(\lambda)\phi_2 + Re(\lambda)\psi_2 + \alpha(x)|\phi - \psi|^2 - f_1\Delta u - f_3\Delta v + f_2\phi + f_4\psi.$$

In the last expression, performing integration and using Green's identity we have

$$Re(\lambda)||\mathbf{U}||^2 + \int_\Omega \alpha(x)|\phi - \psi|^2 dx = \langle \mathbf{U}, \mathbf{F} \rangle,$$

and using Cauchy-Schwartz inequality we obtain

$$Re(\lambda)||\mathbf{U}||^2 + \int_\Omega \alpha(x)|\phi - \psi|^2 dx \leq ||\mathbf{U}||\,||\mathbf{F}||,$$

from where follows that $||(\lambda I - A)^{-1}\mathbf{F}|| \leq C\,||\mathbf{F}||$ with $C = [Re(\lambda)]^{-1}$. The proof of the theorem is complete. $\square$

Resumo. Neste trabalho estabelecemos existência, unicidade e decaimento exponencial de energia para as soluções de um sistema de equações de onda acopladas através de amortecimento distribuído, num domínio suave de qualquer dimensão.

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