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Constructions of Dense Lattices over Number Fields

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ABSTRACT. In this work, we present constructions of algebraic lattices in Euclidean space with optimal center density in dimensions $2,3,4,5,6,8$ and 12, which are rotated versions of the lattices Λ_n, for $n = 2,3,4,5,6,8$ and K_{12}. These algebraic lattices are constructed through canonical homomorphism via $\mathbb{Z}$-modules of the ring of algebraic integers of a number field.

Keywords: algebric lattices, number fields, sphere packings.

1 INTRODUCTION

Algebraic number theory has recently raised a great interest for its new role in algebraic lattice theory and in code design for many different coding applications. Algebraic lattices have been useful in information theory and the question of finding algebraic lattices over number fields maximum center density. The problem of finding algebraic lattices with maximal minimum product distance has been studied in last years and this has motived special attention of many researchs in considering ideals of certain rings. The search for dense algebraic lattices in general dimensions has been encouraged in the last decades because they can be applied to Information Theory [1]- [4].

The classical sphere packing problem consists to find out how densely a large number of identical spheres can be packed together in the Euclidean space. The packing density, $\Delta(\Lambda)$, of a lattice Λ is the proportion of the space $\mathbb{R}^n$ covered by the non-overlapping spheres of maximum radius centered at the points of Λ. The densest possible lattice packings have only be determined in

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dimensions 1 to 8 and 24. It is also known that these densest lattice packings are unique (up to equivalences) [5].

This paper is organized as follows. In Section 2, notions and results from algebraic number theory that are used in the work are reviewed. In Section 3, rotated lattices are constructed from number fields in dimensions $2,3,4,5,6,8$ and 12, which are rotated versions of the lattices Λ_n, for $n = 2,3,4,5,6,8$ and K_{12}.

2 BACKGROUND OF NUMBER FIELDS

Let $\mathbb{K}$ be a number field, i.e., $\mathbb{K}$ is a finite extension of $\mathbb{Q}$. By Primitive Element Theorem, there is an element $\theta \in \mathbb{K}$ such that $\mathbb{K} = \mathbb{Q}(\theta) = \{\sum_{i=0}^{n-1} a_i\theta^i;\ a_i \in \mathbb{Q}\}$, where θ is a root of a polynomial $p(x) \in \mathbb{Q}[x]$ of minimal degree n. A cyclotomic field is a number field such that $\mathbb{K} = \mathbb{Q}(\theta)$, where θ is a primitive n-th root of unity. If $\theta_1 = \theta, \theta_2, \ldots, \theta_n$ are the n distinct roots of $p(x)$, then threre are exactly n distinct $\mathbb{Q}$-embeddings $\sigma_i : \mathbb{K} \to \mathbb{C}$ such that $\sigma_i(\theta) = \theta_i$, for all $i = 1,2,\ldots,n$. Furhtermore, there are r_1 real embeddings $\sigma_1,\ldots,\sigma_{r_1}$ and $2r_2$ complex embeddings $\sigma_{r_1+1}, \overline{\sigma_{r_1+1}}, \ldots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+r_2}}$. If $\Re(x)$ and $\Im(x)$ denote, respectively, the real part and the imaginary part of x, the canonical embedding $\sigma : \mathbb{K} \longrightarrow \mathbb{R}^n$, with $x \in \mathbb{K}$, is defined by

$$\sigma(x) = (\sigma_1(x),\ldots,\sigma_{r_1}(x), \Re(\sigma_{r_1+1}(x)),\ldots,\Im(\sigma_{r_1+r_2}(x)))\,.$$

The set $\mathscr{O}_{\mathbb{K}} = \{\alpha : f(\alpha) = 0$ for some monic polynomial $f(x) \in \mathbb{Z}[x]\}$ is a ring called ring of algebraic integers of $\mathbb{K}$. The ring $\mathscr{O}_{\mathbb{K}}$ has a basis $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ over $\mathbb{Z}$. In other words, every element $\alpha \in \mathscr{O}_{\mathbb{K}}$ is uniquely written as $\alpha = \sum_{i=1}^{n} a_i\alpha_i$, where $a_i \in \mathbb{Z}$ for all $i = 1,2,\ldots,n$, and every nonzero fractional ideal $\mathscr{M}$ of $\mathscr{O}_{\mathbb{K}}$ is a free $\mathbb{Z}$-module of rank n [7].

If $\alpha \in \mathbb{K}$, the value

$$Tr_{\mathbb{K}}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha)$$

is called trace of α in $\mathbb{K}$. If $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ is an integral basis of $\mathbb{K}$, the discriminant of $\mathbb{K}$ is defined as $D_{\mathbb{K}} = \det[\sigma_j(\alpha_i)]^2$ and it is an invariant over change of basis [6].

3 CONSTRUCTIONS OF DENSE ALGEBRAIC LATTICES

A lattice Λ is a discrete additive subgroup of $\mathbb{R}^n$, that is, $\{0\} \neq \Lambda \subseteq \mathbb{R}^n$ is a lattice iff there are linearly independent vectors $\{v_1,v_2,\ldots,v_k\}$, with $k \leq n$, in $\mathbb{R}^n$ such that

$$\Lambda = \{\sum_{i=1}^{k} a_i v_i : a_i \in \mathbb{Z},\ \text{for all } i = 1,2,\ldots,k\}.$$

The set $\{v_1,v_2,\ldots,v_k\}$ is called a basis for Λ, the matrix M whose rows are these vectors is called a generator matrix for Λ and the matrix $G = MM^t$ is called Gram matrix.

If $\mathscr{M}$ is a $\mathbb{Z}$-submodule in $\mathbb{K}$ of rank n, the set $\Lambda = \sigma(\mathscr{M})$ is a lattice in $\mathbb{R}^n$ called an algebraic lattice. The center density of Λ is given by

$$\delta(\Lambda) = \frac{t^{n/2}}{2^n[\mathscr{O}_{\mathbb{K}} : \mathscr{M}]\sqrt{| D_{\mathbb{K}} |}},$$

where $t = \min\{Tr_{\mathbb{K}}(\alpha\overline{\alpha}) : \alpha \in \mathscr{M}, \alpha \neq 0\}$ and $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}]$ denotes the index of the submodule $\mathscr{M}$.

Example 3.1. *If $\mathbb{K} = \mathbb{Q}(\zeta_3)$, where ζ_3 is the primitive 3-th root of unity, then $[\mathbb{K} : \mathbb{Q}] = 2$, $\{1, \zeta_3\}$ is a basis of $\mathbb{K}$ and $D_{\mathbb{K}} = -3$. If $\mathscr{M}$ is a submodule of $\mathscr{O}_{\mathbb{K}}$ given by*

$$\mathscr{M} = \{a_0 + a_1\zeta_3 : a_0, a_1 \in \mathbb{Z}\},$$

then $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 1$ and

$$Tr_{\mathbb{K}}(\alpha\overline{\alpha}) = 2(a_0^2 - a_0a_1 + a_1^2),$$

where $\alpha \in \mathscr{M}$. Since $t = min\{Tr_{\mathbb{K}}(\alpha\overline{\alpha}) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 2$ with $a_0 = 1$ and $a_1 = 0$, it follows that

$$\delta(\mathscr{M}) = \frac{(\sqrt{2}/2)^2}{\sqrt{3}} = \frac{1}{2\sqrt{3}},$$

i.e., the center density of $\sigma(\mathscr{M})$ is the same of the lattice Λ_2. Similarly, if $\mathbb{K} = \mathbb{Q}(\sqrt{3})$, then $[\mathbb{K} : \mathbb{Q}] = 2$, $\{1, \sqrt{3}\}$ is a basis of $\mathbb{K}$ and $D_{\mathbb{K}} = 12$. If $\mathscr{M}$ is a submodule of $\mathscr{O}_{\mathbb{K}}$ given by $\mathscr{M} = \{a_0 + a_1\sqrt{3} : a_0 - a_1 \equiv 0(mod\ 2)$ and $a_0, a_1 \in \mathbb{Z}\}$, then $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 2$ and $Tr_{\mathbb{K}}(\alpha^2) = 8a_0^2 + 24a_0a_1 + 24a_1^2$, where $\alpha \in \mathscr{M}$. Since $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 8$ with $a_0 = 1$ and $a_1 = 0$, it follows that $\delta(\mathscr{M}) = \frac{(\sqrt{8}/2)^2}{2^3\sqrt{3}} = \frac{1}{4\sqrt{3}}$.

Example 3.2. *If $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta = \zeta_9 + \zeta_9^{-1}$ and ζ_9 is the primitive 9-th root of unity, then $[\mathbb{K} : \mathbb{Q}] = 3$, $\{1, \theta, \theta^2\}$ is an integral basis of $\mathbb{K}$ and $D_{\mathbb{K}} = 3^4$. If $\mathscr{M}$ is a submodule of $\mathscr{O}_{\mathbb{K}}$ given by*

$$\mathscr{M} = \{a_0 + a_1\theta + a_2\theta^2 : a_0 \equiv 0(mod\ 2) \text{ and } a_0 + 2a_1 + a_2 \equiv 0(mod\ 3), \text{ where } a_0, a_1, a_2 \in \mathbb{Z}\},$$

then $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 6$ and

$$Tr_{\mathbb{K}}(\alpha^2) = 18(a_0^2 + a_0a_1 + 5a_0a_2 + a_1^2 + 5a_1a_2 + 9a_2^2),$$

where $\alpha \in \mathscr{M}$. Since $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 18$ with $a_0 = 1$ and $a_1 = a_2 = 0$, it follows that

$$\delta(\mathscr{M}) = \frac{(\sqrt{18}/2)^3}{54} = \frac{1}{4\sqrt{2}},$$

i.e., the center density of $\sigma(\mathscr{M})$ is the same of the lattice Λ_3. Similarly, if $\mathscr{M} = \{a_0 + a_1\theta + a_2\theta^2 : a_0 \equiv 0(mod\ 2)$ and $a_0 + 2a_1 + a_2 \equiv 0(mod\ 3)$, where $a_0, a_1, a_2 \in \mathbb{Z}\}$, then $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 6$ and $Tr_{\mathbb{K}}(\alpha^2) = 18(3a_0^2 + 3a_0a_1 + 10a_0a_2 + a_1^2 + 5a_1a_2 + 9a_2^2)$, *where* $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 18$ *with* $a_0 = a_2 = 0$ *and* $a_1 = 1$, *it follows that* $\delta(\mathscr{M}) = \frac{(\sqrt{18}/2)^3}{2 \cdot 3 \cdot 3^2} = \frac{1}{4\sqrt{2}}$. *Similarly, if* $\mathscr{M} = \{a_0 + a_1\theta + a_2\theta^2 : a_0 \equiv 0(mod\ 2)$ *and* $a_0 + 2a_1 + a_2 \equiv 0(mod\ 3)$, *where* $a_0, a_1, a_2 \in \mathbb{Z}\}$, *then* $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 6$ *and* $Tr_{\mathbb{K}}(\alpha^2) = 18(3a_0^2 + 3a_0a_1 + 10a_0a_2 + a_1^2 + 5a_1a_2 + 9a_2^2)$, *where* $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 18$ *with* $a_0 = a_2 = 0$ *and* $a_1 = 1$, *it follows that* $\delta(\mathscr{M}) = \frac{(\sqrt{18}/2)^3}{2 \cdot 3 \cdot 3^2} = \frac{1}{4\sqrt{2}}$. *Finally, if* $\mathbb{K} = \mathbb{Q}(\theta)$, *where* θ *is a root of* $p(x) = x^3 - 3x + 1$, *then* $[\mathbb{K} : \mathbb{Q}] = 3$, $\{1, \theta, \theta^2\}$ *is a basis of* $\mathbb{K}$ *and* $D_{\mathbb{K}} = 3^4$. *If* $\mathscr{M}$ *is a submodule of* $\mathscr{O}_{\mathbb{K}}$ *given by* $\mathscr{M} = \{a_0 + a_1\theta + a_2\theta^2 : a_2 \equiv 0(mod\ 2)$ *and* $a_0 - a_1 + a_2 \equiv 0(mod\ 3)$, *with* $a_0, a_1, a_2 \in \mathbb{Z}\}$, *then* $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 6$ *and* $Tr_{\mathbb{K}}(\alpha^2) = 18(a_0^2 + 5a_0a_1 + 3a_0a_2 + 9a_1^2 + 10a_1a_2 + 3a_2^2)$, *where* $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 18$ *with* $a_0 = 1$ *and* $a_1 = a_2 = 0$, *it follows that* $\delta(\mathscr{M}) = \frac{(\sqrt{18}/2)^3}{2 \cdot 3 \cdot 3^2} = \frac{1}{4\sqrt{2}}$. *Similarly, if* $\mathbb{K} = \mathbb{Q}(\theta)$, *where* $\theta = \zeta_7 + \zeta_7^{-1}$ *and* ζ_7 *is the primitive* 7*-th root of unity, then* $[\mathbb{K} : \mathbb{Q}] = 3$, $\{1, \theta, \theta^2\}$ *is a basis of* $\mathbb{K}$ *and* $D_{\mathbb{K}} = 7^2$. *If* $\mathscr{M}$ *is a submodule of* $\mathscr{O}_{\mathbb{K}}$ *given by* $\mathscr{M} = \{a_0 + a_1\theta + a_2\theta^2 : a_0 \equiv 0(mod\ 7)$ *and* $3a_1 - a_2 \equiv 0(mod\ 7)$, *with* $a_0, a_1, a_2 \in \mathbb{Z}\}$, *then* $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 2\dot{7}^2$ *and* $Tr_{\mathbb{K}}(\alpha^2) = 98(a_0^2 - a_0a_2 + a_1^2 + 7a_1a_2 + 13a_2^2)$, *where* $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 98$ *with* $a_0 = 1$ *and* $a_1 = a_2 = 0$, *it follows that* $\delta(\mathscr{M}) = \frac{(\sqrt{98}/2)^3}{2 \cdot 7^3} = \frac{1}{4\sqrt{2}}$.

Example 3.3. *If* $\mathbb{K} = \mathbb{Q}(\zeta_8)$, *where* ζ_8 *is the primitive* 8*-th root of unity, then* $[\mathbb{K} : \mathbb{Q}] = 4$, $\{1, \zeta_8, \zeta_8^2, \zeta_8^3\}$ *is an integral basis of* $\mathbb{K}$ *and* $D_{\mathbb{K}} = 2^8$. *If* $\mathscr{M}$ *is a submodule of* $\mathscr{O}_{\mathbb{K}}$ *given by*

$$\mathscr{M} = \{a_0 + a_1\zeta_8 + a_2\zeta_8^2 + a_3\zeta_8^3 : a_0 + a_1 + a_2 + a_3 \equiv 0(mod\ 2),\ where\ a_0, a_1, a_2, a_3 \in \mathbb{Z}\},$$

then $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 2$ *and*

$$Tr_{\mathbb{K}}(\alpha\overline{\alpha}) = 8(2a_0^2 - 2a_0a_3 + a_1^2 - a_1a_2 + a2^2 - 2a_2a_3 + 2a_3^2),$$

where $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 8$ *with* $a_1 = 1$ *and* $a_0 = a_2 = a_3 = 0$, *it follows that*

$$\delta(\mathscr{M}) = \frac{(\sqrt{8}/2)^4}{32} = \frac{1}{8},$$

i.e., the center density of $\sigma(\mathscr{M})$ *is the same of the lattice* Λ_4. *Similarly, if* $\mathscr{M} = \{a_0 + a_1\zeta_8 + a_2\zeta_8^2 + a_3\zeta_8^3 : a_2 + a_3 \equiv 0(mod\ 2)$, *where* $a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$, *then* $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 2$ *and* $Tr_{\mathbb{K}}(\alpha^2) = 8(a_0^2 + a_1^2 + a_2^2 + 2a_3^2 + a_0a_2 + 2a_0a_3 + a_1a_2 + 2a_2a_3)$, *where* $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 8$ *with* $a_0 = 1$ *and* $a_1 = a_2 = a_3 = 0$, *it follows that* $\delta(\mathscr{M}) = \frac{(\sqrt{8}/2)^4}{32} = \frac{1}{8}$. *Similarly, if* $\mathbb{K} = \mathbb{Q}(\theta)$, *where* θ *is a root of* $p(x) = x^4 + 3x^2 + 1$, *then* $[\mathbb{K} : \mathbb{Q}] = 4$, *where* $\{1, \theta, \theta^2, \theta^3\}$ *is an integral basis of* $\mathbb{K}$, $D_{\mathbb{K}} = 2^4\dot{5}^2$. *If* $\mathscr{M}$ *is a submodule of* $\mathscr{O}_{\mathbb{K}}$ *given by* $\mathscr{M} = \{a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 : a_0 - 2a_1 + 2a_2 - a_3 \equiv 0(mod\ 5)$, *where* $a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$, *then* $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 10$, *and if* $\alpha \in \mathscr{M}$, *then* $Tr_{\mathbb{K}}(\alpha\overline{\alpha}) = 40a_0^2 - 40a_0a_1 + 132a_0a_2 + 360a_0a_3 + 20a_1^2 - 28a_1a_2 - 140a_1a_3 + 158a_2^2 + 720a_2a_3 + 900a_3^2$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha^2) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 20$ *with* $a_0 = a_2 = a_3 = 0$ *and* $a_1 = 1$, *if follows that* $\delta(\mathscr{M}) = \frac{(\sqrt{20}/2)^4}{2^3 \cdot 5^2} = \frac{1}{8}$.

Example 3.4. *If $\mathbb{K}=\mathbb{Q}(\theta)$, where $\theta=\zeta_{44}^{10}-\zeta_{44}^{12}$ and ζ_{44} is the primitive 44-th root of unity, then $[\mathbb{K}:\mathbb{Q}]=5$, $\{1,\theta,\theta^2,\theta^3,\theta^4\}$ is an integral basis of $\mathbb{K}$ and the discriminant of $\mathbb{K}$ is 11^4. Let $\mathscr{M}$ be a submodule of $\mathscr{O}_{\mathbb{K}}$ given by*

$$\begin{aligned}\mathscr{M} \;=\;& \{a_0+a_1\theta+a_2\theta^2+a_3\theta^3+a_4\theta^4 : a_0\equiv 0(mod\ 11),\ 5a_2+a_3\equiv 0(mod\ 11)\\ & \text{and } a_0+15a_1+11a_2+a_4\equiv 0(mod\ 22),\ \text{where } a_0,a_1,a_2,a_3,a_4\in\mathbb{Z}\}.\end{aligned}$$

In this case, $\mathscr{M}$ is a submodule of $\mathscr{O}_{\mathbb{K}}$ of index $2\cdot 11^3$ and the trace form of $\alpha\in\mathscr{M}$ is given by

$$\begin{aligned}Tr_{\mathbb{K}/\mathbb{Q}}(\alpha^2) \;=\;& 37752a_0^2+43802a_0a_1+79134a_0a_2+16456a_0a_3+136488a_0a_4\\ & +12826a_1^2+46706a_1a_2+10406a_1a_3+79860a_1a_4+44286a_2^2\\ & +26136a_2a_3+144716a_2a_4+9438a_3^2+30976a_3a_4+124388a_4^2.\end{aligned}$$

Thus, $t=min\{Tr_{\mathbb{K}}(\alpha^2):\alpha\in\mathscr{M},\alpha\neq 0\}=242$ with $a_0=a_2=a_3=0, a_1=-3$ and $a_4=1$. As the volume of the lattice $\sigma(mathcalM)$ is $\sqrt{|D_{\mathbb{K}}|}[\mathscr{M}:\mathscr{O}_{\mathbb{K}}]=2\cdot 11^5$, it follows that

$$\delta(\mathscr{M})=\frac{(\sqrt{242}/2)^5}{2\cdot 11^5}=\frac{1}{8\sqrt{2}},$$

i.e., the center density of $\sigma(\mathscr{M})$ is the same of the lattice Λ_7.

Example 3.5. *If $\mathbb{K}=\mathbb{Q}(\zeta_9)$, where ζ_9 is the primitive 9-th root of unity, then $[\mathbb{K}:\mathbb{Q}]=6$, $\{1,\zeta_9,\zeta_9^2,\zeta_9^3,\zeta_9^4,\zeta_9^5\}$ is an integral basis of $\mathbb{K}$ and $D_{\mathbb{K}}=-3^9$. If $\mathscr{M}$ is a submodule of $\mathscr{O}_{\mathbb{K}}$ given by*

$$\begin{aligned}\mathscr{M} \;=\;& \{a_0+a_1\zeta_9+a_2\zeta_9^2+a_3\zeta_9^3+a_4\zeta_9^4+a_5\zeta_9^5 : a_1-a_2+a_4-a_5\equiv 0(mod\ 3),\\ & \text{where } a_0,a_1,\ldots,a_5\in\mathbb{Z}\},\end{aligned}$$

then $[\mathscr{O}_{\mathbb{K}}:\mathscr{M}]=9$ and

$$\begin{aligned}Tr_{\mathbb{K}}(\alpha\overline{\alpha}) \;=\;& 18(a_0^2+a_0a_1+a_0a_2+a_0a_3+2a_0a_4+2a_0a_5+a_1^2+a_1a_3+3a_1a_4\\ & +a_2^2+a_2a_3+3a_2a_5+a_3^2+2a_3a_4+2a_3a_5+3a_4^2+3a_5^2,\end{aligned}$$

where $\alpha\in\mathscr{M}$. Since $t=min\{Tr_{\mathbb{K}}(\alpha\overline{\alpha}):\alpha\in\mathscr{M},\alpha\neq 0\}=18$ with $a_0=1$ and $a_1=a_2=a_3=a_4=a_5=0$, it follows that

$$\delta(\mathscr{M})=\frac{(\sqrt{18}/2)^6}{3^6\sqrt{3}}=\frac{1}{8\sqrt{3}},$$

i.e., the center density of $\sigma(\mathscr{M})$ is the same of the lattice Λ_6.

Example 3.6. *If $\mathbb{K}=\mathbb{Q}(\zeta_{20})$, where ζ_{20} is the primitive 20-th root of unity then $[\mathbb{K}:\mathbb{Q}]=8$, $\{1,\zeta_{20},\zeta_{20}^2,\zeta_{20}^3,\zeta_{20}^4,\zeta_{20}^5,\zeta_{20}^6,\zeta_{20}^7\}$ is an integral basis fo $\mathbb{K}$ and $D_{\mathbb{K}}=2^8\cdot 5^6$. If $\mathscr{M}$ is a submodule of $\mathscr{O}_{\mathbb{K}}$ given by*

$$\begin{aligned}\mathscr{M} \;=\;& \{a_0+a_1\zeta_{20}+a_2\zeta_{20}^2+a_3\zeta_{20}^3+a_4\zeta_{20}^4+a_5\zeta_{20}^5+a_6\zeta_{20}^6+a_7\zeta_{20}^7 : a_0+a_4\equiv 0(mod\ 4),\\ & a_1+a_5\equiv 0(mod\ 2), a_2+a_3+a_6\equiv 0(mod\ 4) \text{ and } a_7\equiv 0(mod\ 5),\ \text{where}\\ & a_0,a_1,\ldots,a_7\in\mathbb{Z}\},\end{aligned}$$

then $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 5$ *and*

$$\begin{aligned} Tr_{\mathbb{K}}(\alpha\overline{\alpha}) &= 20(2a_0^2 + 2a_0a_1 + 5a_0a_2 + 3a_0a_3 + 3a_0a_4 + 2a_0a_5 + 5a_0a_6 + 8a_0a_7 + a_1^2 \\ &\quad +3a_1a_2 + 2a_1a_3 + 2a_1a_4 + a_1a_5 + 3a_1a_6 + 5a_1a_7 + 4a2^2 + 4a_2a_3 + 5a_2a_4 \\ &\quad +3a_2a_5 + 7a_2a_6 + 12a_2a_7 + 2a_3^2 + 3a_3a_4 + 2a_3a_5 + 5a_3a_6 + 7a_3a_7 + 2a_4^2 \\ &\quad +2a_4a_5 + 5a_4a_6 + 8a_4a_7 + a_5^2 + 3a_5a_6 + 5a_5a_7 + 4a_6^2 + 12a_6a_7 + 10a_7^2), \end{aligned}$$

where $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha\overline{\alpha}) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 20$ *with* $a_1 = 1$ *and* $a_0 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0$, *it follows that*

$$\delta(\mathscr{M}) = \frac{(\sqrt{20}/2)^8}{2^4 \cdot 5^4} = \frac{1}{16},$$

i.e., the center density of $\sigma(\mathscr{M})$ *is the same of the lattice* Λ_8.

Example 3.7. *If* $\mathbb{K} = \mathbb{Q}(\zeta_{21})$, *where* ζ_{21} *is the primitive* 21*-th root of unity, then* $[\mathbb{K} : \mathbb{Q}] = 12$, $\{1, \zeta_{21}, \ldots, \zeta_{21}^{11}\}$ *is an integral basis of* $\mathbb{K}$ *and* $D_{\mathbb{K}} = 3^6 \cdot 7^{10}$. *If* $\mathscr{M}$ *is a submodule of* $\mathscr{O}_{\mathbb{K}}$ *given by*

$$\begin{aligned} \mathscr{M} &= (\zeta_{21}^6 - \zeta_{21}^2 + 1)a_0 + (\zeta_{21}^7 - \zeta_{21}^3 + \zeta_{21})a_1 + (\zeta_{21}^8 - \zeta_{21}^4 + \zeta_{21}^2)a_2 \\ &\quad +(\zeta_{21}^9 - \zeta_{21}^5 + \zeta_{21}^3)a_3 + (\zeta_{21}^{10} - \zeta_{21}^6 + \zeta_{21}^4)a_4 + (\zeta_{21}^1 1 - \zeta_{21}^7 + \zeta_{21}^5)a_5 \\ &\quad +(\zeta_{21}^{11} - \zeta_{21}^9 + \zeta_{21}^4 - \zeta_{21}^3 + \zeta_{21} - 1)a_6 \\ &\quad +(\zeta_{21}^{11} - \zeta_{21}^{10} - \zeta_{21}^9 + \zeta_{21}^8 - \zeta_{21}^6 + \zeta_{21}^5 - \zeta_{21}^3 + \zeta_{21}^2 - 1)a_7 \\ &\quad +(-\zeta_{21}^{10} + \zeta_{21}^8 - \zeta_{21}^7 - 1)a_8 + (-\zeta_{21}^{11} + \zeta_{21}^9 - \zeta_{21}^8 - \zeta_{21})a_9 \\ &\quad +(-\zeta_{21}^{11} + \zeta_{21}^{10} - \zeta_{21}^8 + \zeta_{21}^6 - \zeta_{21}^4 + \zeta_{21}^3 - \zeta_{21}^2 - \zeta_{21} + 1)a_{10} \\ &\quad +(-\zeta_{21}^8 + \zeta_{21}^7 + \zeta_{21}^6 - \zeta_{21}^5 - \zeta_{21}^2 + 1)a_{11}, \text{ where } a_0, a_1, \ldots, a_{11} \in \mathbb{Z}\}, \end{aligned}$$

then $[\mathscr{O}_{\mathbb{K}} : \mathscr{M}] = 7$ *and*

$$\begin{aligned} Tr_{\mathbb{K}}(\alpha\overline{\alpha}) &= 28a_0^2 - 14a_0a_2 - 14a_0a_3 - 14a_0a_4 + 28a_0a_5 - 28a_0a_7 - 14a_0a_9 \\ &\quad +28a_0a_{10} + 28a_0a_{11} + 28a_1^2 - 14a_1a_3 - 14a_1a_4 - 14a_1a_5 + 28a_1a_6 \\ &\quad -28a_1a_8 - 14a_1a_{10} + 28a_1a_{11} + 28a_2^2 - 14a_2a_4 - 14a_2a_5 - 14a_2a_6 \\ &\quad +28a_2a_7 - 28a_2a_9 - 14a_2a_{11} + 28a_3^2 - 14a_3a_5 - 14a_3a_6 - 14a_3a_7 \\ &\quad +28a_3a_8 - 28a_3a_{10} + 28a_4^2 - 14a_4a_6 - 14a_4a_7 - 14a_4a_8 + 28a_4a_9 \\ &\quad -28a_4a_{11} + 28a_5^2 - 14a_5a_7 - 14a_5a_8 - 14a_5a_9 + 28a_5a_{10} + 28a_6^2 \\ &\quad -14a_6a_8 - 14a_6a_9 - 14a_6a_{10} + 28a_6a_{11} + 28a_7^2 - 14a_7a_9 - 14a_7a_{10} \\ &\quad -14a_7a_{11} + 28a_8^2 - 14a_8a_{10} - 14a_8a_{11} + 28a_9^2 - 14a_9a_{11} + 28a_{10}^2 \\ &\quad +28a_{11}^2, \end{aligned}$$

where $\alpha \in \mathscr{M}$. *Since* $t = min\{Tr_{\mathbb{K}}(\alpha\overline{\alpha}) : \alpha \in \mathscr{M}, \alpha \neq 0\} = 28$ *with* $a_0 = 1$ *and* $a_1 = a_2 = \cdots = a_{11} = 0$, *it follows that*

$$\delta(\mathscr{M}) = \frac{(\sqrt{28}/2)^{12}}{3^3 \cdot 5^6} = \frac{1}{3^3},$$

i.e., the center density of $\sigma(\mathscr{M})$ *is the same of the lattice* K_{12}.

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RESUMO. Neste trabalho, apresentamos construções de reticulados algébricos no espaço euclidiano com densidade central ótima nas dimensões $2,3,4,5,6,8$ e 12, que são versões rotacionadas dos reticulados Λ_n, para $n = 2,3,4,5,6,8$ e K_{12}, onde esses reticulados algébricos são construídos através do homomorfismo canônico via $\mathbb{Z}$-módulos do anel de inteiros algébricos de um corpo de números.

Palavras-chave: reticulados algébricos, corpos de números, empacotamento esférico.

REFERENCES

[1] A.A. Andrade & R. Palazzo Jr. Linear codes over finite rings. *TEMA - Trends in Applied and Computational Mathematics*, 6(2) (2005), 207-217.

[2] A.S. Ansari, R. Shah, Zia Ur-Rahman & A.A. Andrade. Sequences of primitive and non-primitive BCH codes. *TEMA - Trends in Applied and Computational Mathematics*, 19(2) (2018), 369-389.

[3] A. A. Andrade, A. J. Ferrari & C. W. O. Benedito, Constructions of algebraic lattices, *Comput. Appl. Math.*, **29** (2010) 1-13.

[4] E. Bayer-Fluckiger, Lattices and number fields, In: *Contemp. Math.*, Amer. Math. Soc., Providence (1999), 69-84.

[5] J. H. Conway & N. J. A. Sloane. *Sphere Packings, Lattices and Groups*, 3rd Edition, Springer Verlag, New York (1999).

[6] J. C. Interlando, J. O. D. Lopes & T. P .N. Neto. The discriminant of abelian number fields, *J. Algebra Appl.*, 5 (2006), 35-41.

[7] P. Samuel. *Algebraic Theory of Numbers*, Hermann, Paris (1970).