Tendências em Matemática Aplicada e Computacional, **21**, N. 2 (2020), 359-369

www.scielo.br/tema
doi: 10.5540/tema.2020.021.02.0003590359

On Euler-Lagrange's Equations: A New Approach

G. E. O. GIACAGLIA[1*] and W. Q. LAMAS[2]

Received on October 20, 2019 / Accepted on April 26, 2020

ABSTRACT. A new formalism is proposed to study the dynamics of mechanical systems composed of N connected rigid bodies, by introducing the concept of $6N$-dimensional composed vectors. The approach is based on previous works by the authors where a complete formalism was developed by means of differential geometry, linear algebra, and dynamical systems usual concepts. This new formalism is a method for the description of mechanical systems as a whole and not as each separate part. Euler-Lagrange's Equations are easily obtained by means of this formalism.

Keywords: composed vectors, connected rigid bodies, dynamics.

1 INTRODUCTION

Works by Cortizo and Giacaglia [2], Kottke [8], and Giacaglia and Kottke [5] have described in details the formalism, which is the theoretical background of this work, by the introduction of virtual linear velocities and angular velocities and force-torque composed vectors representing kinematical and dynamical quantities of all links involved.

In the present work we show a compact and straightforward method to obtain Euler-Lagrange's Equations for such a mechanical system by simply writing down the Newton-Euler dynamical equations condensed into composed vectors and then multiplying such equations by a properly chosen composed kinematical vector.

The links are considered to be connected by rotational joints and acted on by arbitrary forces and torques, including torques resulting from frictional forces arising in the joints. The final differential equations are transformed by simple scalar products and it is shown that the resulting differential equations are completely equivalent to the differential equations obtained from the Lagrangian of the system. The method is applicable to connected rigid multi body systems in the

*Corresponding author: G. E. O. Giacaglia – E-mail: giacaglia@gmail.com
[1]Post-graduate Programme in Mechanical Engineering, Department of Mechanical Engineering, University of Taubaté, Rua Daniel Danelli, s/n, 12060-440, Taubaté, SP, Brazil. – E-mail: giacaglia@gmail.com https://orcid.org/0000.0002.1030-4863
[2]Lorena School of Engineering, University of São Paulo, Estrada Municipal do Campinho, s/n, 12602-810, Lorena, SP, Brazil. – E-mail: lamaswq@gmail.com https://orcid.org/0000-0002-7588-0335

presence of holonomic and nonholonomic constraints, a possibility available by the methodology used in the present work [6]. Figure 1 shows an example of a simple system composed by two connected joints, and torque and forces applied.

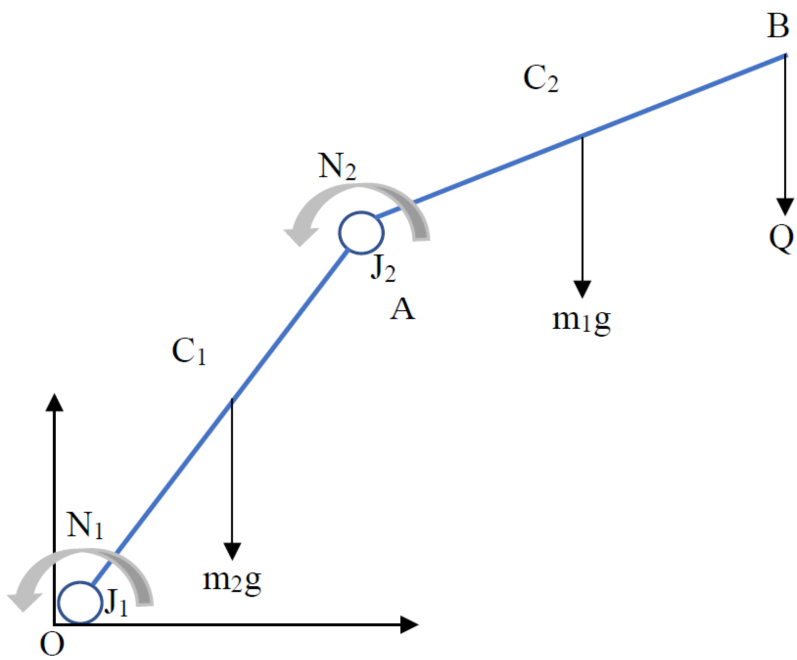

Figure 1: System with two constrained plane links.

Figure 2 gives the geometry of this system, showing the position and orientation of each torque and forces applied.

The first of Figure 2 is the link *OA* and the second is the link *AB*. We have set the lengths to be $2L_2$ and $2L_1$, for simplicity of the geometry and mechanics involved.

Classical work by Roberson [11], Wittenburg [15], and Shabana [14] on multi-bodies systems use classical approaches to this matter, treating each body as a separate entity of these systems. Schiehlen [12] presents a substantial collection of the most efficient algorithms for calculating rigid-body dynamics, but as we understand no composed vectors, suggested here, are used. This is also observed in revised work of this author Schiehlen [13].

A more elaborate algorithm is given by Featherstone [3] where the author informs that rigid body dynamics algorithms presented are the subject of computational rigid-body dynamics through the medium of spatial $6D$ vector notation.

In the present article we actually use an algorithm with $6N$-dimensional vectors, where N is the number of connected bodies involved.

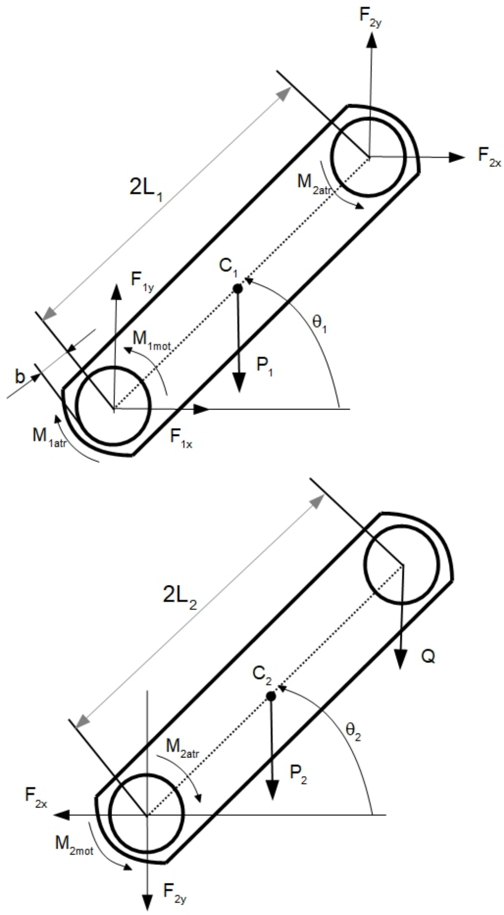

Figure 2: Links geometry and applied forces and torques.

2 DYNAMICS OF THE SYSTEM

We consider a mechanical system constituted by N rigid links connected by rotational joints. The position and orientation of each link is represented by the coordinates of its centre of mass and by a set of three Euler angles each. The number of degrees of freedom of the system, using a set of generalised coordinates, for instance the Denawitt-Hartenberg parameters [7], is represented by $n < N$ *generalised coordinates* $q = (q^1, q^2, \ldots, q^n)$, associated to $\dot{q} = (\dot{q}^1, \ldots, \dot{q}^n)$ *generalised velocities* and $\ddot{q} = (\ddot{q}^1, \ldots, \ddot{q}^n)$ *generalised accelerations* associated to configuration q. The linear velocities of the centre mass of each link and their angular velocities, represented by vectors

$$\vec{v}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^\alpha \vec{v}_{\alpha i}(q)$$

$$\vec{\omega}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^\alpha \vec{\omega}_{\alpha i}(q) \tag{2.1}$$

Vectors $\vec{\upsilon}_{\alpha i}$ and $\vec{\omega}_{\alpha i}$ defined in Eq. (2.1) are the linear and angular velocities of link $i(i=1,\ldots,n)$ in the state where all generalised velocities are zero except the one (α) which is unitary. The absolute value of vector $\vec{\upsilon}_{\alpha i}$ is the distance ℓ_i between point C_i and a reference joint J_i. Vectors $\vec{\omega}_{\alpha i}$ are dimensionless. The linear and angular accelerations are given by

$$\frac{d\vec{\upsilon}_i}{dt}(q,\dot{q},\ddot{q}) = \frac{d}{dt}\left[\sum_{\alpha=1}^{n}\dot{q}^{\alpha}\,\vec{\upsilon}_{\alpha i}(q)\right] = \sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\,\vec{\upsilon}_{\alpha i}(q) + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\,\dot{q}^{\gamma}\frac{\partial\vec{\upsilon}_{\beta i}}{\partial q^{\gamma}(q)} \tag{2.2}$$

$$\frac{d\vec{\omega}_i}{dt}(q,\dot{q},\ddot{q}) = \frac{d}{dt}\left[\sum_{\alpha=1}^{n}\dot{q}^{\alpha}\,\vec{\omega}_{\alpha i}(q)\right] = \sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\,\vec{\omega}_{\alpha i}(q) + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\,\dot{q}^{\gamma}\frac{\partial\vec{\omega}_{\beta i}}{\partial q^{\gamma}(q)} \tag{2.3}$$

The *distributions of actions* on each link is the pair $\left(\vec{F}_i,\vec{N}_i\right)$ and corresponds to all actions (forces and torques) acting on link i, using as pole their centres of mass $C_i(i=1,\ldots,n)$. Euler equations for any particular link i are given by

$$\begin{aligned}
\vec{N}_i(q,\dot{q},\ddot{q}) &= I_i\left(\frac{d\vec{\omega}_i}{dt}\right) + \vec{\omega}_i \times I_i\left(\vec{\omega}_i\right) = \\
&= I_i\left[\sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\,\vec{\omega}_{\alpha i}(q) + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\,\dot{q}^{\gamma}\frac{\partial\vec{\omega}_{\beta i}}{\partial q^{\gamma}}(q)\right] + \\
&+ \left[\sum_{\beta=1}^{n}\dot{q}^{\beta}\,\vec{\omega}_{\beta i}(q)\right] \times I_i\left[\sum_{\gamma=1}^{n}\dot{q}^{\lambda}\,\vec{\omega}_{\gamma i}(q)\right] = \sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\left\{I_i\left[\vec{\omega}_{\alpha i}(q)\right]\right\} + \\
&+ \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\,\dot{q}^{\gamma}\left\{\left[\frac{\partial(I_i\vec{\omega}_{\beta i})}{\partial q^{\gamma}}(q)\right] + \left[\vec{\omega}_{\beta i}(q)\right] \times I_i\left[\vec{\omega}_{\gamma i}(q)\right]\right\}
\end{aligned} \tag{2.4}$$

Newton's Equation for the force acting on the centre of mass C_i of link i is given by

$$\begin{aligned}
\vec{F}_i(q,\dot{q},\ddot{q}) = m_i\left(\frac{d\vec{\upsilon}_i}{dt}\right) &= m_i\left[\sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\,\vec{\upsilon}_{\alpha i}(q) + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\,\dot{q}^{\gamma}\frac{\partial\vec{\upsilon}_{\beta i}}{\partial q^{\gamma}}(q)\right] \\
&= \sum_{\alpha=1}^{n}\ddot{q}^{\alpha}\left[m_i\vec{\upsilon}_{\alpha i}(q)\right] + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\dot{q}^{\beta}\,\dot{q}^{\gamma}\left[m_i\frac{\partial\vec{\upsilon}_{\beta i}}{\partial q^{\gamma}}(q)\right]
\end{aligned} \tag{2.5}$$

The results of the above evaluations corresponding to the $3n$ set $(q,\dot{q},\ddot{q})$ is represented by $\vec{F}_i\,(q,\dot{q},\ddot{q})$ and $\vec{N}_i\,(q,\dot{q},\ddot{q})$ for $i=1,\ldots,n$.

The *composed vector* represented by $\vec{v}\vec{\omega}$ is given by the

$$\vec{v}\vec{\omega} = (\vec{v}_1,\vec{\omega}_1,\vec{v}_2,\vec{\omega}_2,\ldots,\vec{v}_n,\vec{\omega}_n) \tag{2.6}$$

and from the above expressions for the linear and angular velocities it is found that

$$\vec{v}\vec{\omega} = \sum_{\alpha=1}^{n}\dot{q}^{\alpha}\vec{v}\vec{\omega}_{\alpha} \tag{2.7}$$

where

$$\vec{v}\vec{\omega}_\alpha = (\vec{v}_{\alpha 1}, \vec{\omega}_{\alpha 1}, \vec{v}_{\alpha 2}, \vec{\omega}_{\alpha 2}, \ldots, \vec{v}_{\alpha n}, \vec{\omega}_{\alpha n}) \tag{2.8}$$

with components defined as above. The $6n$-dimensional vector $\vec{v}\vec{\omega}$ is the vector of virtual velocities associated to the state $(q, \dot{q})$ of the system.

The pair force-torque is represented by $\left(\vec{F}_i, \vec{N}_i\right), (q, \dot{q}, \ddot{q})$ where the components vectors are given by $\vec{F}_i\,(q, \dot{q}, \ddot{q})$ and $\vec{N}_i\,(q, \dot{q}, \ddot{q})$ with $(i = 1, \ldots, n)$ and corresponding to the $3n$-dimensional set $(q, \dot{q}, \ddot{q})$.

Newton-Euler equations give the resulting forces $\vec{F}_i(q, \dot{q}, \ddot{q})$ and torques $\vec{N}_i(q, \dot{q}, \ddot{q})$ acting on link $i (i = 1, \ldots, N)$ corresponding to the $3n$-dimensional set $(q, \dot{q}, \ddot{q})$.

These $2n$ three dimensional vectors can be composed in only one $6n$-dimensional vector $\vec{F}\vec{N}(q, \dot{q}, \ddot{q})$, given by

$$\vec{F}\vec{N}(q, \dot{q}, \ddot{q}) = \left(\vec{F}_1(q, \dot{q}, \ddot{q}), \vec{N}_1(q, \dot{q}, \ddot{q}), \ldots, \vec{F}_n(q, \dot{q}, \ddot{q}), \vec{N}_n(q, \dot{q}, \ddot{q})\right) \tag{2.9}$$

The dynamical equations for the mechanical system considered are obtained as follows. The expressions derived above (Eqs. (2.4) and Eqs. (2.5)) for $\vec{F}_i(q, \dot{q}, \ddot{q})$ and $\vec{N}_i(q, \dot{q}, \ddot{q})$ can be condensed in a unique composed $6n$-dimensional vector

$$\vec{F}\vec{N}(q, \dot{q}, \ddot{q}) = \sum_{\alpha=1}^{n} \ddot{q}^\alpha \vec{P}\vec{L}_\alpha(q) + \sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} \dot{q}^\beta \dot{q}^\gamma \vec{X}\vec{Y}_{\beta\gamma}(q) \tag{2.10}$$

where

$$\vec{P}\vec{L}_\alpha = \left(\vec{P}_{\alpha 1}, \vec{L}_{\alpha 1}, \vec{P}_{\alpha 2}, \vec{L}_{\alpha 2}, \ldots, \vec{P}_{\alpha n}, \vec{L}_{\alpha n}\right) \tag{2.11}$$

and

$$\vec{X}\vec{Y}_{\beta\gamma} = \left(\vec{X}_{\beta\gamma 1}, \vec{Y}_{\beta\gamma 1}, \vec{X}_{\beta\gamma 2}, \vec{Y}_{\beta\gamma 2}, \ldots, \vec{X}_{\beta\gamma n}, \vec{Y}_{\beta\gamma n}\right) \quad (\alpha, \beta, \gamma = 1, \ldots, n) \tag{2.12}$$

are given by

$$\begin{aligned} \vec{P}_{\alpha i}(q) &= m_i \vec{\upsilon}_{\alpha i}(q) \\ \vec{L}_{\alpha i}(q) &= I_i\,[\vec{\omega}_{\alpha i}(q)] \end{aligned} \tag{2.13}$$

Vectors $\vec{P}_{\alpha i}$ and $\vec{L}_{\alpha i}$ above defined represent the linear and angular momenta of link i in the state of the system where only the α velocity is different from zero and unitary, all other being zero.

We also have that

$$\vec{X}_{\beta\gamma i}(q) = m_i \frac{\partial \vec{\upsilon}_{\beta i}}{\partial q^\gamma}(q) \qquad \vec{Y}_{\beta\gamma i}(q) = I_i \left[\frac{\partial \vec{\omega}_{\beta i}}{\partial q^\gamma}(q)\right] + \left[\vec{\omega}_{\beta i}(q)\right] \times I_i \left[\vec{\omega}_{\gamma i}(q)\right] \tag{2.14}$$

for $i = 1, \ldots, n$.

It should be noted that the scalar product of two composed vectors must satisfy the physical meaning of such product. As an example, consider the power associated to the composed vector

$$\vec{F}\vec{N}\,(q, \dot{q}, \ddot{q}) = \left(\vec{F}_1\,(q, \dot{q}, \ddot{q})\,, \vec{N}_1\,(q, \dot{q}, \ddot{q})\,, \ldots, \vec{F}_n\,(q, \dot{q}, \ddot{q})\,, \vec{N}_n\,(q, \dot{q}, \ddot{q})\right) \tag{2.15}$$

The scalar product of this force-torque action by the linear-angular velocity composed vector $\vec{v}\vec{\omega} = (\vec{v}_1, \vec{\omega}_1, \vec{v}_2, \vec{\omega}_2, \dots, \vec{v}_n, \vec{\omega}_n)$ is given by

$$\vec{F}\vec{N}(q,\dot{q},\ddot{q}) \circ \vec{v}\vec{\omega}(q,\dot{q}) = \sum_{i=1}^{n} \left[\vec{F}_i \circ \vec{v}_i + \vec{N}_i \circ \vec{\omega}_i\right] \tag{2.16}$$

The differential equations of motion of the composed system derived by simple vector algebra give an equivalent result as that given by the explicit form of Euler-Lagrange's Equations [4,10]. In fact, consider the composed vectors, for $\alpha = 1,2,\dots,n$,

$$\vec{v}\vec{\omega}_\alpha = (\vec{v}_{\alpha 1}, \vec{\omega}_{\alpha 1}, \dots, \vec{v}_{\alpha n}, \vec{\omega}_{\alpha n}) \tag{2.17}$$

where vectors $\vec{v}_{\alpha i}, \vec{\omega}_{\alpha i}$ have been defined by Eq. (2.1),

$$\begin{cases} \vec{v}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^\alpha \vec{v}_{\alpha i}(q) \\ \vec{\omega}_i(q,\dot{q}) = \sum_{\alpha=1}^{n} \dot{q}^\alpha \vec{\omega}_{\alpha i}(q) \end{cases} \tag{2.18}$$

Consider Eq. (2.11) and the scalar product

$$\begin{aligned} \vec{P}\vec{L}_\alpha(q) \circ \vec{v}\vec{\omega}_\gamma(q) &= \sum_{i=1}^{n} \left(\vec{P}_{\alpha i} \circ \vec{v}_{\gamma i} + \vec{L}_{\alpha i} \circ \vec{\omega}_{\gamma i}\right) = \\ &= \sum_{i=1}^{n} \left[m_i\left(\vec{v}_{\alpha i} \circ \vec{v}_{\gamma i}\right) + I_i\left(\vec{\omega}_{\alpha i}\right) \circ \vec{\omega}_{\gamma i}\right] = a_{\gamma\alpha}(q) \end{aligned} \tag{2.19}$$

The coefficients $a_{\gamma\alpha}(q)$ are the coefficients of the quadratic terms $\dot{q}^\gamma \dot{q}^\alpha$ in the total kinetic energy of the system, that is

$$T = \frac{1}{2} \sum_\alpha \sum_\beta a_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta \tag{2.20}$$

The first part of Eq. (2.19) on the right hand side is the linear kinetic energy (except for a factor $\frac{\dot{q}^\alpha \dot{q}^\gamma}{2}$) of link i because the velocity of the centre of mass C_i of this link associated to the generalised coordinate (angle) q^α is precisely $\vec{v}_{\alpha i}$ except for a factor $\dot{q}^\alpha$. The second part is the rotational kinetic energy of this same link except for the same factor $\frac{\dot{q}^\alpha \dot{q}^\gamma}{2}$.

The same scalar product is now applied to the second term on the right-hand side of Eq. (2.11), giving

$$\vec{X}\vec{Y}_{\alpha\beta} \circ \vec{v}\vec{\omega}_\gamma(q) = \sum_{i=1}^{n} m_i \frac{\partial \vec{v}_{\alpha i}}{\partial q^\beta} \circ \vec{v}_{\gamma i} + \sum_{i=1}^{n} \left[\frac{\partial I_i\left(\vec{\omega}_{\alpha i}\right)}{\partial q^\beta} + \vec{\omega}_{\alpha i} \times I_i\left(\vec{\omega}_{\beta i}\right)\right] \circ \vec{\omega}_{\gamma i} = a^\gamma_{\beta\alpha}(q) \tag{2.21}$$

The coefficients $a^\gamma_{\beta\alpha}(q)$ correspond to the Christoffel Brackets $\begin{bmatrix} \gamma \\ \beta\alpha \end{bmatrix}$ resulting from the quadratic part of Euler-Lagrange's Equations when the constraints are time independent and considering unitary linear and angular velocities. Taking into account the definition of the coefficients $a_{\gamma\alpha}(q)$ given by Eq. (2.21) and comparing Eq. (2.22) with the classical definition [1,4]

$$\left[\begin{array}{c} i \\ \beta\alpha \end{array}\right] = \frac{\partial a_{i\alpha}}{\partial q^{\beta}} + \frac{\partial a_{i\beta}}{\partial q^{\alpha}} - \frac{\partial a_{\alpha\beta}}{\partial q^{i}} \tag{2.22}$$

it is seen that there is an exact correspondence between the two definitions.

The same scalar product applied to the force-torque composed vector given by Eq. (2.16), gives

$$\vec{F}\vec{N}\,(q,\dot{q},\ddot{q}) \circ \vec{v}\vec{\omega}_{\gamma}(q) = \sum_{i=1}^{n} \vec{F}_i \circ \vec{v}_{\gamma i} + \sum_{i=1}^{n} \vec{N}_i \circ \vec{\omega}_{\gamma i} = Q_{\gamma} \tag{2.23}$$

Quantities Q_γ are the generalised forces and give a direct physical interpretation of these quantities. From Eq. (2.23) the dimension of the generalised forces associated to an angular generalised coordinate q^α shows that it corresponds to the torque of the applied force with respect to joint J_i.

With the above definitions, the equations of motion for the generalised coordinates assume the known form

$$\sum_{\alpha} a_{\gamma\alpha}\ddot{q}^{\alpha} + \sum_{\alpha}\sum_{\beta} \left[\begin{array}{c} \gamma \\ \beta\alpha \end{array}\right] \dot{q}^{\alpha}\dot{q}^{\beta} = Q_{\gamma} \tag{2.24}$$

From Eq. (2.24) we obtain the explicit form of Euler-Lagrange's Equations for a system with time independent constraints given by

$$\ddot{q}^{s} + \sum_{\alpha}\sum_{\beta} \left(\begin{array}{c} s \\ \beta\alpha \end{array}\right) \dot{q}^{\alpha}\dot{q}^{\beta} = Q^{s} \tag{2.25}$$

where we have used the usual Christoffel Parentheses [9] and the generalised forces Q^s corresponding to the action on the generalised coordinate q^s.

The above development shows how the use of composed vectors leads to Euler-Lagrange's Equations of motion of a system composed of connected rigid bodies. The computation of all quantities involved in Eq. (2.19) is straightforward from the computation of forces and torques acting on the system.

3 APPLICATION: TWO-LINK ARTICULATED ARM

A simple example will be used in order to follow the mathematical formalism without complicating the physical system. We consider the simple two-link system shown in Figure 1 and representing two articulated links on a fixed base.

The rotational joint J_1 is articulated to a fixed support. The rotational joint J_2 is connecting links OA and AB whose lengths are L_1 and $2L_2$. The system moves on a fixed vertical plane. The generalised coordinates are the angles $\theta_1 = q^1$ e $\theta_2 = q^2$, that links OA and AB make with a fixed horizontal line, so that its time derivatives are the absolute angular velocities of each link. This is done instead of defining the angle $\theta = \theta_2 - \theta_1$ between links OB and AO, as it is done in usual robots' kinematics. For the purpose of this example there is no point in adopting this notation. The positions of points C_1 and C_2 are given by

$$C_1 - O = L_1 \left(c_1 \vec{e}_1 + s_1 \vec{e}_2\right) \tag{3.1}$$

$$C_2 - O = 2L_1 \left(c_1 \vec{e}_1 + s_1 \hat{j}\right) + L_2 \left(c_2 \vec{e}_1 + s_2 \hat{j}\right) \tag{3.2}$$

where $(\vec{e}_1, \vec{e}_2)$ is an inertial base and $c_1 = cosq^1$, $s_1 = sinq^1$, $c_2 = cosq^2$ and $s_2 = sinq^2$.

The linear velocities of these points are

$$\begin{aligned}
\vec{v}_1 &= \dot{q}^1 \vec{v}_{11} + \dot{q}^2 \vec{v}_{21} \\
\vec{v}_{11} &= L_1 \left(-s_1 \vec{e}_1 + c_1 \vec{e}_2\right) \\
\vec{v}_{21} &= \vec{0} \\
\vec{v}_2 &= \dot{q}^1 \vec{v}_{21} + \dot{q}^2 \vec{v}_{22} \\
\vec{v}_{12} &= 2L_1 \left(-s_1 \vec{e}_1 + c_1 \vec{e}_2\right) \\
\vec{v}_{22} &= L_2 \left(-s_2 \vec{e}_1 + c_2 \vec{e}_2\right)
\end{aligned} \tag{3.3}$$

The angular velocities of each link are

$$\begin{aligned}
\vec{\omega}_1 &= \dot{q}^1 \vec{\omega}_{11} \\
\vec{\omega}_2 &= \dot{q}^2 \vec{\omega}_{22}
\end{aligned} \tag{3.4}$$

where

$$\vec{\omega}_{11} = \vec{\omega}_{22} = \vec{e}_3 \tag{3.5}$$

since angles have been chosen to be with respect to an inertial line.

The accelerations of these points are also easily found

$$\begin{aligned}
\frac{d\vec{v}_1}{dt} &= \ddot{q}^1 \bar{v}_{11} + \dot{q}^1 \dot{q}^1 \vec{a}_{11} \\
\vec{a}_{11} &= -L_1 \left(c_1 \vec{e}_1 + s_1 \vec{e}_2\right) \\
\frac{d\vec{v}_2}{dt} &= \ddot{q}^1 \bar{v}_{12} + \ddot{q}^2 \vec{v}_{22} + \dot{q}^1 \dot{q}^1 \vec{a}_{12} + \dot{q}^2 \dot{q}^2 \vec{a}_{22} \\
\vec{a}_{12} &= -2L_1 \left(c_1 \vec{e}_1 + s_1 \vec{e}_2\right) \\
\vec{a}_{22} &= -L_2 \left(c_2 \vec{e}_1 + s_2 \vec{e}_2\right) \\
\dot{\vec{\omega}}_1 &= \ddot{q}^1 \vec{e}_3 \\
\dot{\vec{\omega}}_2 &= \ddot{q}^2 \vec{e}_3
\end{aligned} \tag{3.6}$$

For the example of Figure 1 it is found that

$$\begin{aligned}
\vec{F}_1 \left(q, \dot{q}, \ddot{q}\right) &= -m_1 g \hat{e}_2 \\
\vec{F}_2 \left(q, \dot{q}, \ddot{q}\right) &= -m_2 g \hat{e}_2 \\
\vec{N}_1 \left(q, \dot{q}, \ddot{q}\right) &= N_1 \vec{e}_3 \\
\vec{N}_2 \left(q, \dot{q}, \ddot{q}\right) &= N_2 \vec{e}_3
\end{aligned} \tag{3.7}$$

In the equation

$$\overrightarrow{FN}(q,\dot{q},\ddot{q}) = \sum_{\alpha=1}^{2} \ddot{q}^{\alpha}\overrightarrow{PL}_{\alpha}(q) + \sum_{\beta=1}^{2}\sum_{\gamma=1}^{2} \dot{q}^{\beta}\dot{q}^{\gamma}\overrightarrow{XY}_{\beta\gamma}(q) \tag{3.8}$$

we find that

$$\begin{aligned} \overrightarrow{PL}_1 &= \left[m_1\vec{v}_{11}\left(q^1\right), I_1\vec{e}_3, m_2\vec{v}_{12}\left(q^1\right), \vec{0}\right] \\ \overrightarrow{PL}_2 &= \left[m_2\vec{v}_{21}\left(q^1\right), \vec{0}, m_2\vec{v}_{22}\left(q^1\right), I_2\vec{e}_3\right] \\ \overrightarrow{XY}_{11}(q) &= \left[m_1\vec{a}_{11}\left(q^1\right), \vec{0}, m_2\vec{a}_{12}\left(q^1\right), \vec{0}\right] \\ \overrightarrow{XY}_{22}(q) &= \left[\vec{0}, \vec{0}, m_2\vec{a}_{22}\left(q^2\right), \vec{0}\right] \end{aligned} \tag{3.9}$$

and all other composed vectors being zero.

The differential equation of motion is

$$\overrightarrow{FN}\left(\vec{F}_1, \vec{N}_1, \vec{F}_2, \vec{N}_2\right) = \ddot{q}^1\overrightarrow{PL}_1 + \ddot{q}^2\overrightarrow{PL}_2 + \dot{q}^1\dot{q}^1\overrightarrow{XY}_{11} + \dot{q}^2\dot{q}^2\overrightarrow{XY}_{22} \tag{3.10}$$

with the above definitions for the coefficients of this equation.

We now consider the composed vectors

$$\begin{aligned} \overrightarrow{v\omega}_1 &= \left[\vec{v}_{11}\left(q^1\right), \vec{e}_3, \vec{v}_{12}\left(q^1\right), \vec{0}\right] \\ \overrightarrow{v\omega}_2 &= \left[\vec{v}_{21}\left(q^1\right), \vec{0}, \vec{v}_{22}\left(q^1\right), \vec{e}_3\right] \end{aligned} \tag{3.11}$$

It is easily found that

$$\begin{aligned} \overrightarrow{FN}\circ\overrightarrow{v\omega}_1 &= -\left(m_1+2m_2\right)gl_1c_1 + N_1 = Q_1 \\ \overrightarrow{FN}\circ\overrightarrow{v\omega}_2 &= -m_2gl_2c_2 + N_2 = Q_2 \end{aligned} \tag{3.12}$$

It is also easily found that

$$\begin{aligned} \overrightarrow{PL}_1\circ\overrightarrow{v\omega}_1 &= \left(m_1+4m_2\right)l_1^2 + I_1 = a_{11}(q) \\ \overrightarrow{PL}_1\circ\overrightarrow{v\omega}_2 &= 2l_1l_2m_2c_{12} \\ c_{12} &= \cos\left(q^1-q^2\right) = a_{21} \\ \overrightarrow{PL}_2\circ\overrightarrow{v\omega}_1 &= 2m_2l_1l_2c_{12} = a_{12} \\ \overrightarrow{PL}_2\circ\overrightarrow{v\omega}_2 &= m_2l_2^2 + I_2 = a_{22} \end{aligned} \tag{3.13}$$

From the definition of the Christoffel Brackets, in this particular example it is found that the only non-zero bracket is

$$\begin{aligned} a_{22}^1 &= 2m_2l_1l_2s_{12} \\ s_{12} &= \sin\left(q^1-q^2\right) \end{aligned} \tag{3.14}$$

The differential equations are given by

$$\left[(m_1+4m_2)\,l_1^2+I_1\right]\ddot{q}^1+[2m_2l_1l_2c_12]\,\ddot{q}^2+[2m_2l_1l_2s_{12}]\,\dot{q}^2\dot{q}^2=(m_1+2m_2)\,gl_1c_1+N_1 \quad (3.15)$$

$$[2l_1l_2m_2c_{12}]\,\ddot{q}^1+\left[m_2l_2^2+I_2\right]\ddot{q}^2=-m_2gl_2c_2+N_2 \quad (3.16)$$

Collecting constant coefficients

$$A_1\ddot{q}^1+B_1\cos\left(q^1-q^2\right)\ddot{q}^2+C_1\sin\left(q^1-q^2\right)\dot{q}^2\dot{q}^2=D_1\cos q^1N_1 \quad (3.17)$$

$$A_2\cos\left(q^1-q^2\right)\ddot{q}^1+B_2\ddot{q}^2=C_2+D_2\cos q^2+N_2 \quad (3.18)$$

In order to have explicit equations for $\ddot{q}^1$ and for $\ddot{q}^2$ it is a simple matter of solving the above system for these two quantities. In this simple example, expressing $\ddot{q}^1$ in terms of all other terms in Eq. (3.18) and substituting into Eq. (3.17) we obtain a differential equation containing only the first and second derivatives of q^2 and the first derivative of q^1 with coefficients functions of both variables q^1 and q^2. Obviously, this is the well-known problem of a double pendulum with the additional complication of the presence of torques applied to the two arms of the pendulum.

4 CONCLUSION

It has been shown that the use of composed linear-angular velocities and force-torque vectors, a new method of representing kinematical and dynamical quantities, is a very efficient and compact form to derive the dynamical equations of multi-body, as compared with classical approaches.

ACKNOWLEDGEMENTS

Dr. Giorgio Eugenio Oscare Giacaglia thanks his ”Productivity Scholarship in Technological Development and Innovative Extension”, supported by the Brazilian National Council for Scientific and Technological Development (CNPq), grant number 314689/2018-4.

Dr. Wendell de Queiróz Lamas thanks his ”Productivity Scholarship in Research”, supported by the Brazilian National Council for Scientific and Technological Development (CNPq), grant number 300992/2018-1.

RESUMO. Um novo formalismo é proposto para estudar a dinâmica de sistemas mecânicos compostos por N corpos rígidos conectados, introduzindo o conceito de vetores compostos 6N-dimensionais. A abordagem é baseada em trabalhos anteriores dos autores, nos quais um formalismo completo foi desenvolvido por meio dos conceitos usuais de geometria diferencial, álgebra linear e sistemas dinâmicos. Esse novo formalismo é um método para a descrição de sistemas mecânicos como um todo e não como cada parte separada. As equações de Euler-Lagrange são facilmente obtidas por meio desse formalismo.

Palavras-chave: vetores compostos, corpos rígidos conectados, dinâmica.

REFERENCES

[1] R. Abraham & J.E. Marsden. "Foundations of Mechanics". American Mathematical Society (AMS), Providence, RI (2008).

[2] S.F. Cortizo & G.E.O. Giacaglia. Dynamics of Multibody: A Geometric Approach (1993).

[3] R. Featherstone. "Rigid Body Dynamics Algorithms". Springer US, Boston, MA (2008).

[4] G.E.O. Giacaglia. "Mecânica Analítica". Almeida Neves, Rio de Janeiro, RJ (1977).

[5] G.E.O. Giacaglia & H.C. Kottke. "The Newton-Euler Multibody Equations Revisited". Associação Brasileira de Engenharia e Ciências Mecânicas (ABCM), Rio de Janeiro, RJ (2007).

[6] G.E.O. Giacaglia & W.Q. Lamas. Notes on Newton-Euler formulation of robotic manipulators. *Proc. Inst. Mech Eng Pt K-J Multi-Body Dyn.*, **226** (2012), 61–71.

[7] A. Goldenberg, B. Benhabib & R. Fenton. A complete generalized solution to the inverse kinematics of robots. *IEEE J. Robot. Autom.*, **1** (1985), 14–20.

[8] H.C. Kottke. Uma Visão Global de Newton-Euler Aplicada à Robótica (2006).

[9] P. Libermann & C.M. Marle. "Symplectic Geometry and Analytical Mechanics". Springer Netherlands, Dordrecht (2012).

[10] J.E. Marsden & J. Scheurle. The reduced Euler-Lagrange equations. *Fields Inst. Comm.*, **1** (1993), 139–164.

[11] R.E. Roberson & R. Schawertassek. "Dynamics of Multibody Systems". Springer-Verlag, Berlin (1988).

[12] W. Schiehlen. "Multibody Systems Handbook". Springer-Verlag, Berlin (1990).

[13] W. Schiehlen & P. Eberhard. "Technische Dynamik". Springer-Verlag, Wiesbaden (2017).

[14] A.A. Shabana. "Dynamics of Multibody Systems". Cambridge University Press, Cambridge (2013).

[15] J. Wittenburg. "Dynamics of Systems of Rigid Bodies". Vieweg+Teubner Verlag, Wiesbaden (1977).