

Power Series Solution for Korteweg-De Vries Equation

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ABSTRACT. We apply the similarity method to the Korteweg-de Vries equation, where we obtain a new equation, in terms of similarity variable. We use the power series method, getting the similarity solution, which is exemplified graphically by particular cases.

Keywords: KdV equation, similarity method, power series solution.

1 INTRODUCTION

Historically, the Korteweg-de Vries equation has its mark from an observation made by engineer J.S. Russell in 1834, but it was only reported in 1844, in his article entitled “ Report on waves”. His experiments aroused great interest in the subject of water waves, but their results received strong criticism from G.B. Airy (1845) and G.G. Stokes (1847). However, in 1985 D. Korteweg and G. de Vries presented a consistent mathematical model, which interpreted the problem initially presented by J.S. Russell. They showed an analytical solution to the problem, which results in the wave observed by Russell, known today as soliton, resulting from the balance between dispersion and the nonlinear effects of this phenomenon [2, 5].

This work is developed as follows: In section 2, we present the Korteweg-de Vries equation and its properties. In section 3, we present a special case of the similarity method, which is the Lie group of scale transformation, in order to find the similarity variable, which allows us to obtain the similarity equation, an ordinary differential equation, in terms of this new variable. In section 4, we present similarity solution, using the power series methodology, concluding with graphics of the particular cases. Finally, we present our considerations.

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2 KORTEWEG-DE VRIES EQUATION

The Korteweg-de Vries (KdV) equation is defined by:

$$\frac{\partial}{\partial t}\eta(x,t) + 6\eta(x,t)\frac{\partial}{\partial x}\eta(x,t) + \frac{\partial^3}{\partial x^3}\eta(x,t) = 0. \quad (2.1)$$

It is mainly responsible for the generation and propagation of shallow water waves, among several other phenomena, in which a small non linearity is combined with a dispersion. The KdV equation arises in many physical problems, which include water waves of long wavelengths and plasma waves [1, 5].

3 LIE GROUP SCALE TRANSFORMATION

Therefore, in order to obtain a solution to the equation (2.1), we will apply a particular case of similarity method [1, 7, 8], which belongs to the scale transformation Lie group, where it allows find invariant solutions in relation to the transformation group and physically interpret the problem.

It is important observe that the symmetry of Lie point

$$(t, x, \eta) \mapsto (\lambda^h \bar{t}, \lambda \bar{x}, \lambda^l \bar{\eta}) \quad (3.1)$$

is the application of the infinitesimal generator associate of the equation (2.1):

$$X \equiv \tau(t, x, \eta)\partial_t + \xi(x, t, \eta)\partial_x + \varphi(t, x, \eta)\partial_\eta \quad (3.2)$$

In order to implement the symmetry algorithm we need to calculate the third order prolongation of the field vector field (3.2) [3] by:

$$Pr^{(3)}X \equiv X + \varphi^t \partial_{\eta_t} + \varphi^x \partial_{\eta_x} + \varphi^{xx} \partial_{\eta_{xx}} + \varphi^{xxx} \partial_{\eta_{xxx}} \quad (3.3)$$

where

$$\begin{aligned} \varphi^t &= D_t \varphi - \eta_t D_t \tau - \eta_x D_t \xi, \\ \varphi^x &= D_x \varphi - \eta_t D_x \tau - \eta_x D_x \xi, \\ \varphi^{xx} &= D_x \varphi^x - \eta_{xt} D_x \tau - \eta_{xx} D_x \xi, \\ \varphi^{xxx} &= D_x \varphi^{xx} - \eta_{xxt} D_x \tau - \eta_{xxx} D_x \xi. \end{aligned}$$

calculating the next equation

$$Pr^{(3)}X \equiv 0, \quad (3.4)$$

we find the determinant equations and then the invariant solutions for equation (2.1). More precisely, we find the scale transformations (3.1).

We define the following transformation group:

$$T_\lambda : \lambda \bar{x} = x; \quad \lambda^{h+l} \bar{t} = t; \quad \lambda^l \bar{\eta} = \eta, \tag{3.5}$$

Replacing the transformation group T_λ , defined by equation (3.5), in the equation (2.1), we have:

$$\lambda^{-h+l} \frac{d}{d\bar{t}} \bar{\eta} + 6x^{l-1+l} \bar{\eta} \frac{d}{d\bar{x}} \bar{\eta} + \lambda^{-3+l} \frac{d^3}{d\bar{x}^3} \bar{\eta} = 0,$$

imposing the invariant condition, we get the values $h = 3$ and $l = -2$. We apply the values in equation 3.5 and build the similarity solution:

$$\eta(x, t) = (3t)^{-\frac{2}{3}} F(\xi) \text{ com } \xi = x(3t)^{-\frac{1}{3}}. \tag{3.6}$$

replacing $\eta(x, t)$ given by equation 3.6 in the equation 2.1:

$$\frac{d}{dt} \left[(3t)^{-\frac{2}{3}} F(\xi) \right] + 6 \left[(3t)^{-\frac{2}{3}} F(\xi) \right] \frac{d}{dx} \left[(3t)^{-\frac{2}{3}} \right] + \frac{d^3}{dx^3} \left[(3t)^{-\frac{2}{3}} F(\xi) \right], \tag{3.7}$$

we apply the chain rule and after some simplifications, we get:

$$\left[-2F(\xi) - \xi \frac{d}{d\xi} F(\xi) F(\xi) \right] + 6F(\xi) \frac{d}{d\xi} + \frac{d^3}{d\xi^3} F(\xi) = 0,$$

we rewrite in this form:

$$F'''(\xi) + (6F(\xi) - \xi) F'(\xi) - 2F(\xi) = 0. \tag{3.8}$$

4 ANALYTICAL SOLUTION IN TERMS OF THE POWER SERIES

In general, the solution of differential equations in terms of power series it is quite common in solving linear differential equations, because when we apply the power series in an ordinary differential equation, we fall into a relationship linear recurrence, which has several works involving its resolution. In the nonlinear case, we are faced with nonlinear recurrences, which increases the complexity in solving and understanding the problem [4, 6]. In this section, let's solve the equation (3.8), we suppose the solution in terms the power series:

$$F(\xi) = \sum_{n=0}^{\infty} C_n \xi^n, \tag{4.1}$$

we replace $F(\xi)$ given by equation (4.1) in equation (3.8), that is,

$$\sum_{k=3}^{\infty} k(k-1)(k-2) C_k \xi^{k-3} + \left(6 \sum_{k=0}^{\infty} C_k \xi^k - \xi \right) \sum_{k=1}^{\infty} k C_k \xi^{k-1} - 2 \sum_{k=0}^{\infty} C_k \xi^k = 0$$

$$\sum_{k=3}^{\infty} k(k-1)(k-2) C_k \xi^{k-3} + 6 \sum_{k=0}^{\infty} C_k \xi^k \sum_{k=1}^{\infty} k C_k \xi^{k-1} - \sum_{k=1}^{\infty} k C_k \xi^k - 2 \sum_{k=0}^{\infty} C_k \xi^k = 0 \tag{4.2}$$

we can get:

$$\sum_{k=0}^{\infty} (k+3)(k+2)(k+1)C_{k+3}\xi^k + 6 \sum_{k=0}^{\infty} C_k \xi^k \sum_{k=0}^{\infty} (k+1)C_{k+1}\xi^k - \sum_{k=0}^{\infty} kC_k \xi^k - 2 \sum_{k=0}^{\infty} C_k \xi^k = 0$$

$$\sum_{k=0}^{\infty} [(k+3)(k+2)(k+1)C_{k+3} - kC_k - 2C_k]\xi^k = -6 \sum_{k=0}^{\infty} C_k \xi^k \sum_{k=0}^{\infty} (k+1)C_{k+1}\xi^k, \quad (4.3)$$

which gives us the nonlinear recurrence relation:

$$(k+3)(k+2)(k+1)C_{k+3} - kC_k - 2C_k = -6 \sum_{i=0}^k C_i(k-i+1)C_{k-i+1}$$

$$C_{k+3} = \frac{C_k}{(k+3)(k+1)} - \frac{6}{(k+3)(k+2)(k+1)} \sum_{i=0}^k C_i(k-i+1)C_{k-i+1} \quad (4.4)$$

Therefore, the exact solution of the equation (2.1) is defined by:

$$\eta(x,t) = \sum_{k=0}^{\infty} C_k x^k (3t)^{-\frac{k+2}{3}}, \quad (4.5)$$

where the values C_k is defined by equation (4.4). The coefficients C_0, C_1 and C_2 are determined for initial conditions to get a particular solution for equation (3.8) and consequently we find a particular solution to equation (2.1). As example, we consider the numeric values to C_0, C_1 and C_2 , plotting the graphics to equation (4.5).

5 CONCLUSIONS

Figure 1 corresponds to the graph of $\eta(x,t)$ for the coefficients $C_0 = 1.5, C_1 = 0.2$ and $C_2 = -0.1$. In it we can see that the peak value is given at the initial instant and then the distribution is stabilized to zero. We observe that the graph of this figure has a symmetry relation with the plane formed by the t axes and the vertical axis that represents the values of $\eta(x,t)$ and perpendicular to the x axis at the point $x = 0$. In Figure 2 we also have a peak at the initial time, but not at the extremes, as in Figure 1, with the distribution reaching zero from the time $t = 0.5$. We also add that the particular solutions are decaying and with a value of $\eta(x,t)$ going to 0, value reached at the instant $t = 0.5$. We emphasize the general solution given by the equation 4.5, allows us to investigate more properties for the KdV equation and thus have a better understanding of its applications in real problems. Therefore, the presented method represents a new way of approaching problems described by the KdV equation in addition to those that are represented by traveling waves.

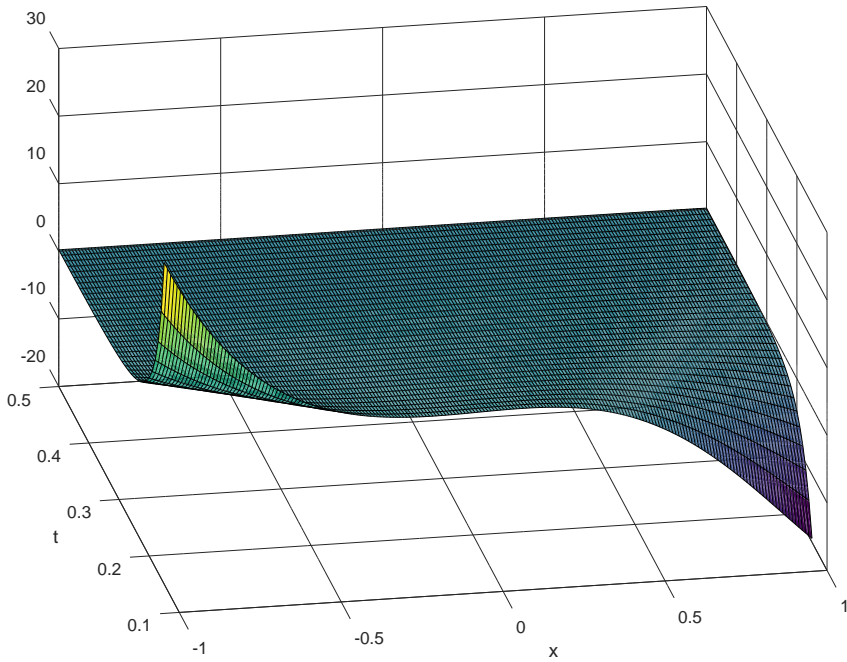


Figure 1: Graphic of $\eta(x,t)$, with $C_0 = 1.5, C_1 = 0.2$ and $C_2 = -0.1$.

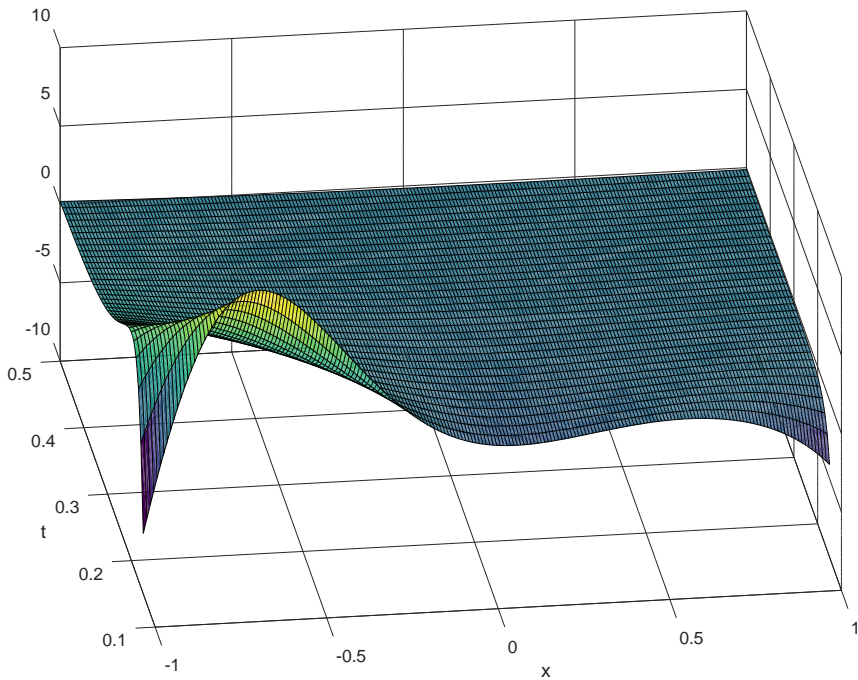


Figure 2: Graphic of $\eta(x,t)$, with $C_0 = -1, C_1 = -0.5$ and $C_2 = 1$.

REFERENCES

- [1] P.L. de Sousa. “Lie symmetries of Korteweg-de Vries equation with non linear dispersion”. Master thesis, Física-UFSCar, São Carlos (2015).
- [2] L. Debnath. “Nonlinear Partial Differential Equations for Scientists and Engineers”. Birkhäuser, 3 ed. (2012).
- [3] F. Güngör, V. Lahno & R. Zhdanov. Symmetry classification of KdV-type nonlinear evolution equations. *Journal of Mathematical Physics*, **45**(6) (2004), 2280–2313.
- [4] A. Hussain, S. Bano, I. Khan, D. Baleanu & K.S. Nisar. Lie Symmetry Analysis, Explicit Solutions and Conservation Laws of a Spatially Two-Dimensional Burgers-Huxley Equation. *Symmetry, MDI*, **12**(170) (2020).
- [5] A. Kasman. A brief history of solitons and the KdV equation. *Current Science*, **115**(8) (2018), 1487–1496.
- [6] E. López-Sandoval, A. Mello, J.J. Godina-Nava & A.R. Samana. Power Series Solution for Solving Nonlinear Burgers-Type Equations. *Abstract and Applied Analysis*, **2015**(ID 712584) (2015), 9 pages.
- [7] C.A. Soares. “Lie symmetries of Burgers equation generalized”. Master thesis, Imecc, Campinas-SP (2011).
- [8] J.C.A. Soares. “Fractional calculus and evolution equations”. Phd thesis, Imecc, Campinas (2016).

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