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Global Stability for SIR Models with Nonlinear Incidence and Removal Functions

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ABSTRACT. In this work, we provide some sufficient conditions to study the global asymptotic stability of the endemic equilibrium for certain models in mathematical epidemiology with nonlinear incidence and removal functions. We also present numerical examples in order to illustrate our results.

Keywords: global asymptotic stability, Hurwitz matrix, epidemiology models.

1 INTRODUCTION

Forecasting the evolution of an infectious disease has been the main motivation for the construction of mathematical epidemiological models. This is because knowing the evolution of the infectious disease allows the design of public health strategies to control or eradicate the disease. In epidemiology, many mathematical models descend from the classical SIR model of Kermack and McKendrick, established in 1927. When the infectious diseases confer permanent acquired immunity, these diseases can be modeled by classical susceptible-infectious-recovered (SIR) models. In epidemiological models the total population, $N(t)$, is divided into any number of classes according to their epidemiological status. In the SIR model, S is the number of individuals in the susceptible class, I is the number of individuals who are infectious but not isolated and R is the number of individuals who are recovered.

We propose the following SIR model

$$\begin{aligned} \dot{S} &= \mu N - \mu S - f(S,I), \\ \dot{I} &= f(S,I) - (\gamma+\mu)I - g_1(I), \\ \dot{R} &= \gamma I + g_1(I) - \mu R, \end{aligned} \tag{1.1}$$

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with $N(t) = S(t) + I(t) + R(t)$. In model (1.1), it has an inflow of newborns into the susceptible class at rate μN and deaths in the classes at rates μS, μI and μR. Notice that, The births balance the deaths. Therefore, the population size N is constant. Also, it is assumed that the infected population recovers at a rate γ and joins the recovered class. The interaction between susceptible and infected population will produce new infected individuals. These contagion processes are characterized by the C^1 incidence function $f(S,I)$. Define $g(I) := (\mu + \gamma)I + g_1(I)$, for some $g_1(I) \geq 0$, we make the following assumptions:

i) $f(S,0) = f(0,I) = 0$ and $f(S,I)$ is a positive function for $S,I > 0$.

ii) $f(S,I)$ is monotonically growing for $S > 0$.

iii) $g(0) = 0$ and $g'(I) > 0$ for $I \geq 0$.

The asymptotic behavior of SIR models with the general nonlinear incidence rate have been studied by many researchers; see [1,2,6,7] and [3]. However, the treatment rate of the infectious is assumed to be linear i.e., $g_1(I) = 0$. In [8,9] a SIR model with a saturated treatment is studied i.e., $g_1(I) := \frac{rI}{1+\alpha I}$, nevertheless it deals only with a specific incidence rate. The general strategy in the previous works has been, first to establish the existence of an endemic point, then to prove that it is unique and later to proof the global stability in the feasibility region.

In this work, we develop criteria on global stability without the need to prove the existence and uniqueness of an endemic point, our result will also allow us to recover several results in the literature such as those above mentioned and extend them for general removal terms.

2 RESULTS

Without loss of generality, since the total population $N(t)$ is constant, we take $N(t) = 1$. Then $1 = S(t) + I(t) + R(t)$. To analyze the endemic points, we reduce model (1.1) to a two-dimensional system as R does not appear in the first two equations of (1.1), the third equation can be ignored. This observation gives the simpler system

$$\begin{aligned} \dot{S} &= \mu - \mu S - f(S,I), \\ \dot{I} &= f(S,I) - g(I). \end{aligned} \tag{2.1}$$

By hypotheses i) and iii), we have that (2.1) always admits the disease-free equilibrium state $E_0 = (S_0, I_0) = (1, 0)$.

In the following result, we prove that all solutions of system (2.1) are eventually confined in the a compact subset, which is a positive invariant region.

Lemma 1. *The region*

$$\Delta = \{(S,I) \in \mathbb{R}^2 : S \geq 0, I \geq 0, S + I \leq 1\}$$

is a positive invariant region for system (2.1).

Proof. Consider a solution of system (2.1), given by $(S(t), I(t))$, with initial condition $(S(0), I(0)) \in \Delta$. We define $w(t) = S(t) + I(t)$. Using equations given in (2.1), it is obtained

$$\dot{w} = \mu - \mu w + f(S,I) - f(S,I) - \gamma I - g_1(I) \leq \mu - \mu w. \tag{2.2}$$

Multiplying expression (2.2) by $e^{\mu t}$ leads to

$$e^{\mu t}\dot{w} + \mu e^{\mu t} w \leq \mu e^{\mu t}. \tag{2.3}$$

Notice that, equation (2.3) can be written as

$$\dot{e^{\mu t} w} \leq \dot{e^{\mu t}}. \tag{2.4}$$

By integrating (2.4) in $[0,t]$ and since $w(0) = S(0) + I(0) \leq 1$, then

$$e^{\mu t} w(t) \leq e^{\mu t} - 1 + w(0) \leq e^{\mu t}. \tag{2.5}$$

Therefore.

$$w(t) \leq 1. \tag{2.6}$$

That is, a solution $(S(t), I(t))$ of system (2.1) does not come out on the side $S + I = 1$.

On the other hand, consider the initial condition $(S(0), I(0)) = (0, I(0)) \in \Delta$ with $I(0) > 0$. Then, the dynamics of system (2.1) in the vertical side of Δ is given by

$$\begin{aligned} \dot{S} &= \mu, \\ \dot{I} &= -g(I(0)). \end{aligned} \tag{2.7}$$

From the reduced system given by (2.7), it is shown that every solution with initial condition $(S(0), I(0)) = (0, I(0))$ goes into Δ. Therefore, these solutions do not come out for the vertical side of Δ.

Finally, we consider a solution of system (2.1) on the horizontal side of Δ. That is, $(S(0), I(0)) = (S(0), 0)$ with $S(0) \geq 0$. Thus, system (2.1) constrained to this type of initial condition is given by

$$\begin{aligned} \dot{S} &= \mu(1 - S), \\ \dot{I} &= 0. \end{aligned} \tag{2.8}$$

From (2.8), it is concluded that every solution of the model with an initial condition $(S(0), I(0)) = (S(0), 0)$ remains on the horizontal side of Δ. Therefore, these solutions do not come out for the horizontal side of Δ.

Therefore, all solutions in Δ remain in this subset when $t \to \infty$. In conclusion, Δ is a positive invariant region. □

From Lemma 1, it can be concluded that every solution of system (2.1) in the first quadrant will eventually enter or remain in Δ.

Let Δ^o denote the interior of Δ.

Recall that a real matrix is Hurwitz if its eigenvalues have negative real parts. In dimension 2, the condition that a matrix A is Hurwitz is equivalent to requiring $Trace(A) < 0$ and $Det(A) > 0$.

The basic reproduction number $\mathscr{R}_0$ has been defined as the average number of secondary infections that occur when one infective is introduced into a completely susceptible host population [1]. To calculate the basic reproduction number, we use the method proposes in [1]. Then, for system (2.1) , the basic reproduction number is

$$\mathscr{R}_0 = \frac{1}{g'(0)} \frac{\partial f(E_0)}{\partial I}.$$

We denote by X the vector field formed by the right hand side of system (2.1). We get

Lemma 2. *Let $f(S,I)$ and $g(I)$ be functions satisfying i)-iii). If $\mathscr{R}_0 > 1$, then E_0 is saddle for X.*

Proof. In effect, we have that E_0 is the equilibrium point, by i) we get $\frac{\partial f(E_0)}{\partial S} = 0$ and since $R_0 > 1$ then $\frac{\partial f(E_0)}{\partial I} - g'(I_0) > 0$. Hence the derivative of X in E_0 is

$$D(X)(E_0) = \begin{pmatrix} -\mu & -\frac{\partial f(E_0)}{\partial I} \\ & \\ 0 & \frac{\partial f(E_0)}{\partial I} - g'(I_0) \end{pmatrix}$$

which has eigenvalues of different signs, this concludes the proof. □

Let us introduce the following notation.

Definition 1. *For C^1-functions on an open set $U \subset \mathbb{R}^2$, $f_1, f_2 : U \to \mathbb{R}$, with independent variables x, y, we consider the* partial Wronskian with respect to *y, as*

$$W_y(f_1, f_2) := \det \begin{pmatrix} f_1 & \frac{\partial f_1}{\partial y} \\ & \\ f_2 & \frac{\partial f_2}{\partial y} \end{pmatrix} = f_1 \frac{\partial f_2}{\partial y} - f_2 \frac{\partial f_1}{\partial y}.$$

Theorem 3. *Let $f(S,I)$ and $g(I)$ be functions satisfying i)-iii). If $\mathscr{R}_0 > 1$ and*

$$W_I(f, g) \geq 0, \tag{2.9}$$

then (2.1) *admits a unique endemic equilibrium, which is globally asymptotically stable relative to Δ^o.*

Proof. We denote by X the vector field formed by the right hand side of system (2.1) and consider the vector field $\frac{1}{g}X$. Notice that $\frac{1}{g}X$ and X have the same phase portraits in Δ^o. By a straightforward computation and (2.9) we obtain

$$\text{Trace}\left(D\left(\frac{1}{g}X\right)\right) = -\frac{1}{g^2}\left[g\left(\mu + \frac{\partial f}{\partial S}\right) + W_I(f,g)\right] < 0. \tag{2.10}$$

Therefore by Bendixson-Dulac's criterion, there are no periodic solutions or polycycles in Δ^o [4].

Since Δ^o is a positive invariant region for $\frac{1}{g}X$, the Poincaré-Bendixson Theorem implies that for any $x \in \Delta^o$ its ω-limit set is a fixed point which cannot be E_0 because this is saddle i.e, $\omega(x) = \{p_\lambda\}$, $p_\lambda \in \Delta^o$, for some index, $\lambda \in \Lambda$. Notice that, Λ is the set of indexes that records the number of equilibria points of the system. This implies the existence of at least one point of endemic equilibrium.

By direct computation and (2.9) we have

$$\text{Det}\left(D\left(\frac{1}{g}X\right)\right) = \frac{1}{g^3}\left[\mu W_I(f,g) + \frac{\partial f}{\partial S}(\mu - \mu S)g'\right] > 0. \tag{2.11}$$

Thus by (2.10) and (2.11) the Jacobian matrix of $\frac{1}{g}X(p)$ is Hurwitz for all $p \in \Delta^o$. Therefore, the fixed points p_λ are isolated and asymptotically stable.

Now let

$$W^s_{p_\lambda} := \{x \in \Delta^o : \omega(x) = \{p_\lambda\}\} \tag{2.12}$$

be the basin of attraction for each p_λ. These sets are open and not empty. As $\Delta^o = \cup_{\lambda\in\Lambda} W^s_{p_\lambda}$ and Δ^o is connected , then it is obtained that the cardinality of Λ is one, which is denoted by $\#(\Lambda) = 1$. Therefore there is a unique endemic equilibrium E^*, which is asymptotically stable and all solutions with initial data in Δ^o converge to E^*. This finishes the proof. □

Recall that a function $k(S,I)$ on Δ^o is *uniformly sublinear function* if

$$\frac{\partial k}{\partial I}(S,I) \leq \frac{k(S,I)}{I} \text{ for all } 0 < S, I < 1. \tag{2.13}$$

As was observed by [2], if a C^2-function $k(S,I)$ is concave with respect to the variable I *i.e.*, $\frac{\partial^2 k}{\partial^2 I} \leq 0$, then it is uniformly sublinear.

As a direct consequence of Theorem 2.2, we have the following statement.

Corollary 4. *Let $f(S,I)$ and $g(I)$ be functions satisfying i)-iii). If $\mathscr{R}_0 > 1$ and if $-g$, f, are uniformly sublinear functions, then* (2.1) *admits a unique endemic equilibrium, which is globally asymptotically stable relative to Δ^o.*

Proof. By the sublinearity of f and $-g$ we have $\frac{\partial f}{\partial I} \leq \frac{f}{I}$ and $\frac{\partial g}{\partial I} \geq \frac{g}{I}$ respectively, combining these inequalities we get

$$W_I(f,g) = f\frac{\partial g}{\partial I} - g\frac{\partial f}{\partial I} \geq f\frac{g}{I} - g\frac{f}{I} = 0,$$

this yields the condition (2.9) as desired. The result follows from Theorem 2.2. □

An immediate consequence of Corollary 2.3 and the remarks above is the following result.

Corollary 5. *Let $f(S,I)$ and $g(I)$ be functions satisfying i)-iii). If $\mathscr{R}_0 > 1$, g is convex and f concave with respect to I, then* (2.1) *admits a unique endemic equilibrium, which is globally asymptotically stable relative to Δ^o.* The Theorem 2.2 extends and/or refines certain previously established results. It is shown in [2], Theorem 2.6 that if f grows monotonically with respect to both variables, is uniformly sublinear. If in addition $\mathscr{R}_0 > 1$ and $g(I) = (\mu+\gamma)I$, then system (2.1) has a unique positive endemic equilibrium state which is globally asymptotically stable. Notice that inequality (2.9) is valid if f is uniformly sublinear and if $g(I) = (\mu+\gamma)I$. Therefore our Theorem 2.2 recovers that result of global stability.

On the other hand, for f concave with respect to I and $g(I) = (\mu+\gamma)I$ a similar result on global stability was established in [6], Theorem 2.1. As we mentioned if f is concave respect to I, then f is uniformly sublinear. So Theorem 2.2 also recovers that result of global stability.

2.1 Examples

In this section, we explore epidemiological models with nonlinear incidence and removal terms. Using our previous results, we analyze the global stability of the endemic equilibrium. To understand our results more intuitively, some numerical simulations are also carried out.

Example 2.1. *Consider the SIR model determined by*

$$\begin{cases} \dot{S} &= \mu - \mu S - \frac{\beta SI}{1+\alpha I}, \\ \dot{I} &= \frac{\beta SI}{1+\alpha I} - (\mu+\gamma)I - r_1 I^{r_2} - s_1 I^{s_2}, \end{cases} \tag{2.14}$$

where $r_2, s_2 \geq 1$; $\gamma, \alpha, r_1, s_1 \geq 0$ and $\beta, \mu > 0$. Since $g(I) = (\mu+\gamma)I + r_1 I^{r_2} + s_1 I^{s_2}$ is convex and $f(S,I)$ concave with respect to I, then by Corollary 2.4 there is a unique endemic equilibrium which is globally asymptotically stable whenever $\mathscr{R}_0 > 1$. Taking the SIR model (2.14) *with specific parameters $\mu = 0.3, \gamma = 0.1, r_1 = 0.2, r_2 = 2, s_1 = 0.3, s_2 = 3, \alpha = 1$ and $\beta = 3$, we get $\mathscr{R}_0 = 7.5$. Hence system* (2.14) *admits a unique endemic equilibrium which is globally stable. See Figure 1.*

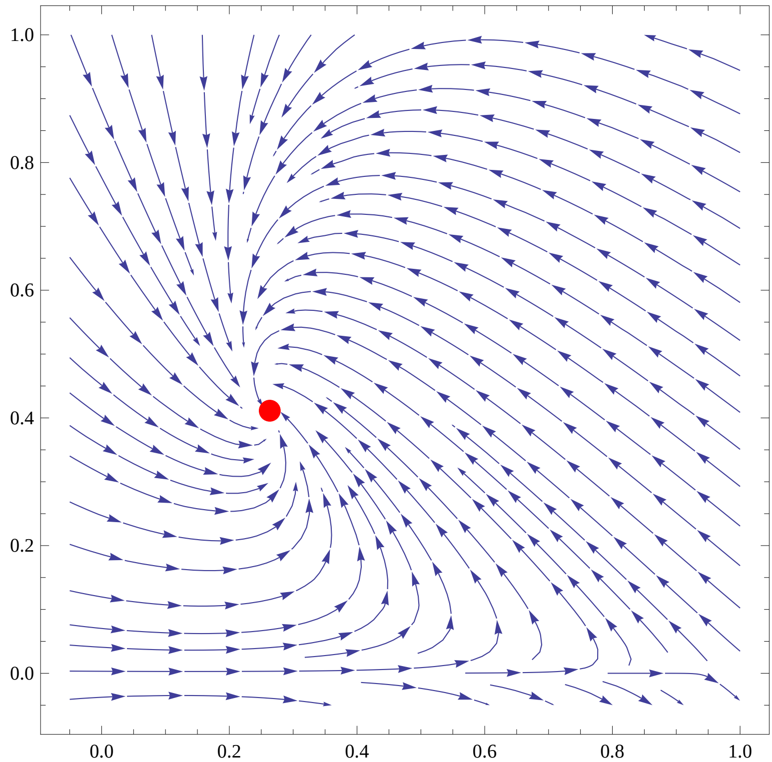

Figure 1: Figure illustrates the dynamic behavior of system (2.14) in the positive quadrant of (S,I) plane. In this scenario, the function $f(S,I) = \frac{\beta SI}{1+\alpha I}$ shows a saturation phenomenon in I while $g(I) = (\mu+\gamma)I + r_1 I^{r_2} + s_1 I^{s_2}$ is always increasing in the variable I.

Example 2.2. *Take for example the SIR model given by*

$$\begin{cases} \dot{S} &= \mu - \mu S - \frac{\beta SI}{1+\alpha I}, \\ \dot{I} &= \frac{\beta SI}{1+\alpha I} - (\mu+\gamma)I - \frac{rI}{1+kI}, \end{cases} \tag{2.15}$$

where all parameters are positive. If $\alpha \geq k$ and $\mathscr{R}_0 > 1$, then there is a unique endemic equilibrium which is globally asymptotically stable. Indeed, a straightforward computation yields $W_I(f,g) \geq 0$ on Δ^o. Therefore, we can apply Theorem 2.2. In particular, for the system (2.15) *with specific parameters $\mu = 0.25, \gamma = 0.6, r = 0.7, k = 0.1, \alpha = 0.4$ and $\beta = 2.5$, we get $\mathscr{R}_0 \approx 1.6129$. So* (2.15) *admits a unique endemic equilibrium which is globally stable. See Figure 2. Notice that in this example g is not convex.*

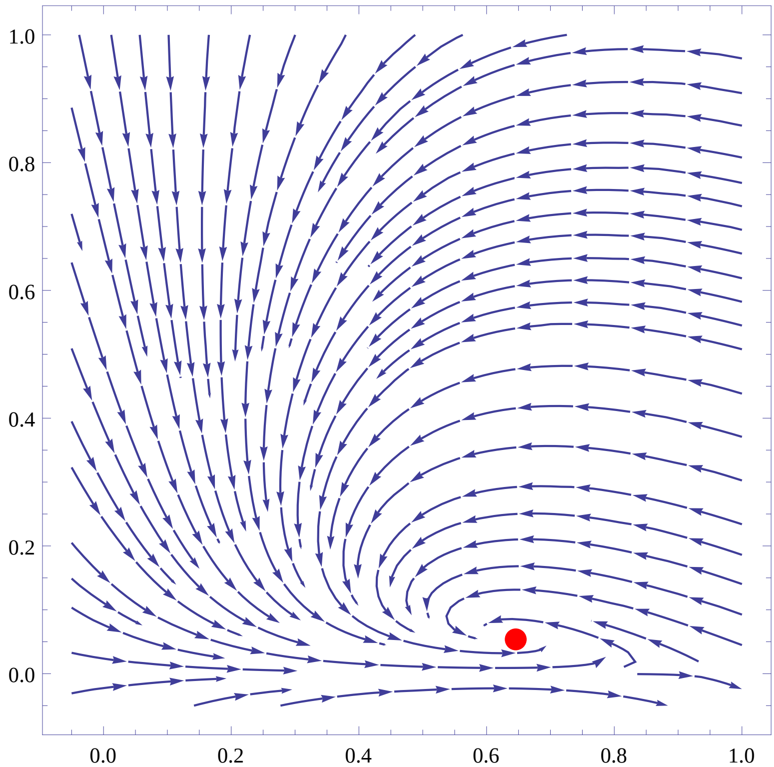

Figure 2: Figure shows the existence of a global attractor of model (2.15). In this scenario, a saturation phenomenon, as a function of I, is considered in the number of new infections and in the number of recovered individuals.

Example 2.3. *We consider the following SIR model with incidence rate proposed in [5] and a similar removal function* $g(I)$*,*

$$\begin{cases} \dot{S} &= \mu - \mu S - \beta S e^{-mI} I, \\ \dot{I} &= \beta S e^{-mI} I - (\mu+\gamma) I + r e^{-kI} I, \end{cases} \tag{2.16}$$

with all parameters are positive. If $m \geq k$ *and* $\mathscr{R}_0 > 1$*, then there is a unique endemic equilibrium which is globally asymptotically stable. Indeed a direct computation gives*

$$W_I(f,g) \geq r\beta S I^2 e^{-(m+k)I}(m-k) \geq 0.$$

On the other hand, the monotonic growing condition iii), $g'(I) \geq 0$*, is equivalent to* $(\mu+\gamma)e^{kI} + rkI \geq r$*, which is true for* $0 \leq I \leq 1$*, due to* $\mu+\gamma > r$*. The last inequality is implied by* $\mathscr{R}_0 > 1$*. Therefore, we can apply Theorem 2.2. In particular, for the system* (2.16) *with parameters* $\mu = 0.4, \gamma = 0.6, r = 0.2, m = 3, k = 1$ *and* $\beta = 20$*, we get* $\mathscr{R}_0 \approx 16.6667$*. Thus, system* (2.16) *admits a unique endemic equilibrium which is globally stable. See Figure 3. Notice that in this example* f *is not monotone with respect to* I*.*

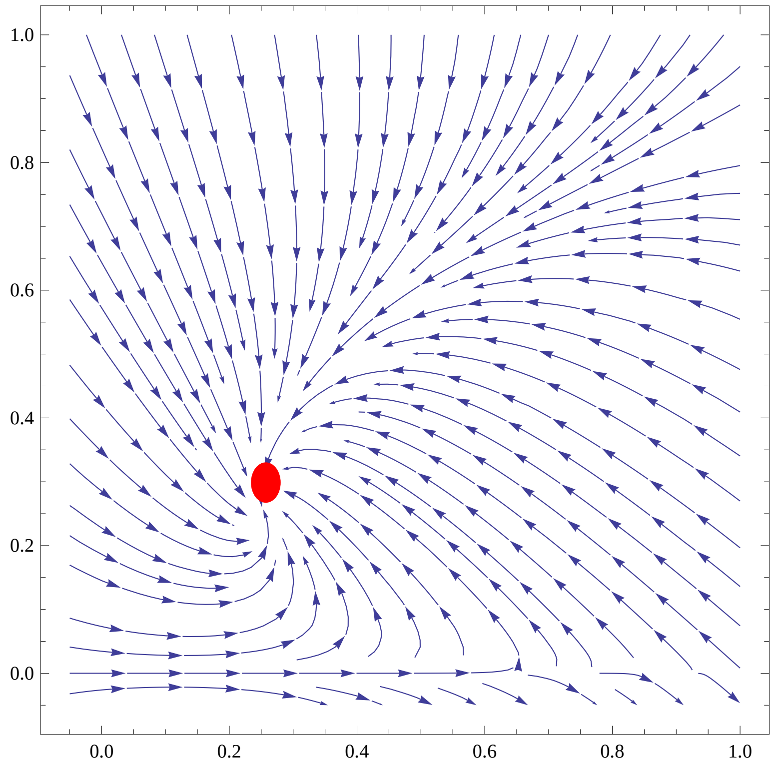

Figure 3: Figure shows numerical simulations of solutions of model (2.16) when the functions $f(S,I)$ and $g(I)$ satisfy the conditions i)-iii) and $\mathscr{R}_0 > 1$. Notice that, when the number of infectious individuals increases, the incidence function $f(S,I) = \beta S e^{-mI} I$ and the number of recovered individuals, $g(I) = re^{-kI} I$ are non monotone functions. However, the solutions of the model converge to the global attractor.

3 DISCUSSION

In this paper, a new coordinate-free approach was proposed to establish the existence, the uniqueness and the global stability of an endemic equilibrium point for SIR models. These criteria recover several results in the literature. We also provided numerical examples in order to illustrate our results. As future work, we intend to study new sufficient conditions of type (2.9) that cover new situations or other models.

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