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Linear Multistep Methods and Order Stars: Some Properties

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Abstract: Order stars theory, introduced by Wanner *et al* (1978), have become a fundamental tool for understanding of order and stability properties of numerical methods. In this work we intend to study some properties of numerical linear multistep methods using this theory.

Keywords: Order stars, stability, numerical methods.

1. Introduction

Order stars theory was first introduced by Wanner [6] and has become a fundamental tool in numerical analysis applied to differential equations. The great idea behind it is to study the behavior of numerical methods through the properties of analytical functions. In this way the order of a numerical method may be see as an exponential approximation problem. This theory set up relation between order and stability of a numerical method and help us to choose a better computer method. In this work we present a study of some properties and definitions from orders stars theory applied on the proof of the *First Dahlquist Barrier* to the order of a good numerical linear multistep method to solve an ordinary differential equation. We begin with a brief presentation of the order stars theory.

1.1. Order Stars of First Kind

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We say that f is an *essential-analytic function* if it is a meromorphic function, that is, if it is an analytic function except in poles and a finite number of essential singularities in the closed complex plane given by $cl\mathbb{C} = \mathbb{C} \cup \{\infty\}$. Remember that a point $z_o \in \mathbb{C} \cup \{\infty\}$ is called a pole of order m if

$$f(z) = \sum_{j=0}^{m} \frac{\beta_j}{(z-z_o)^j} + \sum_{l=0}^{\infty} \alpha_j (z-z_o)^j, \quad 1 \le m < \infty,$$

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and an essential-singularity when $m = \infty$.
Assume that f is an essential-analytic function being approximated by a rational function R given by

$$R(z) = \sum_{j=0}^{m} a_j z^j / \sum_{l=0}^{n} b_l z^l.$$

Excluding the trivial case $f \equiv R$ we define

$$\rho(z) = R(z)/f(z), \ \ z \in cl\mathbb{C}.$$

We can see that ρ is an essential-analytic function. Furthermore its interpolation points, that is, the points z_o for which $f(z_o) = R(z_o)$, are explicit in the expression of ρ being equivalent to $\rho(z_o) = 1$. With this terminology we can now give the first formally definition.

Definition 1.1. *The first kind order star $\{f, R\}$ is defined as a partition of the closed complex plane as $\{A_-, A_0, A_+\}$ where*

$$A_- = \{z \in cl\mathbb{C} : |\rho(z)| < 1\},$$

$$A_0 = \{z \in cl\mathbb{C} : |\rho(z)| = 1\}$$

$$A_+ = \{z \in cl\mathbb{C} : |\rho(z)| > 1\}.$$

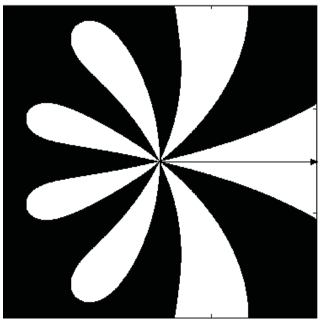

Figure 1: First kind order star $\{e^z, 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^6}{6!}\}$, A_+ in black.

It is clear that A_0 is the boundary of A_+ and A_- and that it is the union of simple Jordan curves (see Figure (1)). We are interested in obtain information about the interpolation properties, stability, location of zeroes and the behavior nearly essential-singularities from the order stars geometry.

Definition 1.2. *Let z_o be an element of $cl\mathbb{C}$, f be an analytical function on neighborhood of it and R a rational function. If there exists a constant $C \neq 0$ and an integer $p \geq 1$ such that*

$$\begin{cases} |z_o| < \infty & \Rightarrow \quad R(z) = f(z) + C(z - z_o)^p + O(|z - z_o|^{p+1}) \\ z_o = \infty & \Rightarrow \quad R(z) = f(z) + Cz^{-p} + O(|z|^{-p-1}) \end{cases},$$

we say that z_o is an interpolation point of order p.

Clearly z_o belongs to A_0. Furthermore we can read the interpolation order of z_o from the order star geometry by counting the numbers of sector of A_- and A_+ close to it.

Definition 1.3. *We define the index of a point z_o by $\iota(z_o)$ as the number sectors of A_- that are close to it (this definition needs more rigor when z_o is an essential singularity [6]).*

Definition 1.4. *Let $z_o \in A_0$ and $p = \iota(z_o) > 0$. We say that z_o is a regular point when ρ is analytic in z_o and there are exactly p sectors of A_- and p sectors of A_+ close to it, with an asymptotic angle of π/p.*

Theorem 1.1. *Let z_o be a zero of f with multiplicity $k \geq 0$. If it is an interpolation point of order $p \geq \max\{1,\, k+1\}$ then $\iota(z_o) = p - k$ and it is regular.*

Proof: We may assume, with no lost of generality, that $z_0 \neq \infty$. Thus

$$f(z) = C_1(z - z_o)^k + O(|z - z_o|^{k+1}), \quad C_1 \neq 0,$$

and follows from the definition of ρ that

$$\rho(z) = 1 + (C/C_1)(z - z_o)^{p-k} + O(|z - z_o|^{p-k+1}), \quad C \neq 0.$$

Let $s = p - k$, $C_2 = C/C_1 \neq 0$. We can take $z = z_o + re^{i\,\theta}$, $r > 0$, to obtain

$$|\rho(z)|^2 = 1 + r^s Re\{C_2 e^{i\,s\,\theta}\} + O(r^{s+1}) = \phi(r, \theta).$$

Given $\epsilon > 0$ and take r sufficiently small, it is clear that $z \in A_+$ when $Re\{C_2 e^{i\,s\,\theta}\} > \epsilon$ and that $z \in A_-$ when $Re\{C_2 e^{i\,s\,\theta}\} < -\epsilon$. Since $Re\{C_2 e^{i\,s\,\theta}\}$ changes of signal $2s$ times to equally spaced values of $\theta \in [0, 2\pi]$ the theorem is proved, except to the cases in which there are sectors of A_+ or A_- with angle asymptotic null (cusps). To complete the proof we will show that this case does not occurs. Cusps may occur just if, when $r \to 0$, there exists distinct values of θ such that $\phi(r, \theta) = 1$ converging to the same value. This implies that a zero of $d\phi(r, \theta)/d\theta$ tends to a zero of $\phi(r, \theta)$, which is a contradiction since ρ is analytic in an neighborhood of z_o, $d\phi(r, \theta)/d\theta = -srIm\{C_2 e^{i\,s\,\theta}\}$ and $Re\{C_2 e^{i\,s\,\theta}\}$, $Im\{C_2 e^{i\,s\,\theta}\} = 0$ does not coexist because $C_2 \neq 0$. Theorem is then proved. □

We will establish a relationship between the location of the zeros and poles of ρ and the interpolation points. The connected components of A_+ and A_- are called the A_+-regions and the A_--regions, respectively. Such regions are called analytical if ρ is analytical on its boundaries. Remembering that the interpolation points belong to A_0, the common points of the boundary of A_+ and A_-. We say that an A_+-region or an A_--region is of multiplicity L when its oriented boundary contain exactly L interpolation points (note that this points need not be necessarily distinct).

Theorem 1.2. *The multiplicity of an analytical A_+-region is the number of poles of ρ, counting with its multiplicities into the domain, and $1 \leq L < \infty$. An equivalent condition hold for an analytical A_--region.*

Proof: Let U be an analytical A_+-region of multiplicity L. We will consider the oriented boundary of U by parameters as $\gamma(t) = \gamma_R(t) + i\gamma_I(t)$, $0 \leq t \leq 1$, where both γ_R and γ_I are real functions. Since ρ is analytical on ∂U and U is a level set of $|\rho|$ we can see that γ is analytical in [0, 1], except in a finite number of points. Let $v(t) = (\gamma_R'(t),\, \gamma_I'(t))$ e $n(t) = (\gamma_I'(t),\, -\gamma_R'(t))$ the tangent and the normal vector to $\gamma(t)$, respectively. From the definition of A_+ we have that when we stay close to ∂U inside the domain, $|\rho|$ decreases locally. Thus $\ln|\rho|$ decrease too along n, close to ∂U. Writing ρ in polar coordinates as

$$\rho(z) = r(x,y)e^{i\,\phi(x,y)}, \quad z = x + i\,y \in cl\mathbb{C},$$

it follows that $\partial \ln r(x,y)/\partial n < 0$. Cauchy-Riemann conditions applied to ρ along γ implies that $\partial \ln r/\partial x = \partial\phi/\partial y$ and $\partial \ln r/\partial y = -\partial\phi/\partial x$. Therefore

$$\frac{\partial \ln y}{\partial n} = \gamma_I' \frac{\partial \ln r}{\partial x} - \gamma_R' \frac{\partial \ln r}{\partial y} = \gamma_I' \frac{\partial \phi}{\partial y} - \gamma_R' \left(-\frac{\partial \phi}{\partial x} \right) = \frac{\partial \phi}{\partial v}$$

and follows that arg ρ decreases ever γ is analytic. As we know ρ is meromorphic in U and γ is the union of closed curves on $cl\mathbb{C}$. Hence, by the argument principle, the variation $\arg\rho$ along γ is $-2\pi L_1$, where L_1 is the number of poles in U (remember that the definition of A_+ implies that ρ has no zeros in U). Furthermore, the interpolation points on γ are the same point in which $\rho = 1$, thus $arg\,\rho$ is an integer multiple of 2π, precisely L_1 times, for $t \in [0,\, 1)$, proving that L_1 (the number of poles) is the multiplicity L of the A_+-region U.
The inequality $L \geq 1$ is another consequence of the fact that arg ρ decreases along the boundary of U. Furthermore $L < \infty$ because, otherwise, there exists infinities poles in U with accumulation points in its boundary. Since an accumulation point of poles is an essential singularity this cases does note occurs because ρ is analytic along ∂U. The proof to an A_--region follows in the same way. □

1.2. Order Stars of Second Kind

Let f be a complex function and R an approximation. We define $\tilde{\rho}(z) = R(z) - f(z)$ for all $z \in cl\mathbb{C}$ where f and R are well defined.

Definition 1.5. *The second kind order star of f and R is a partition of $cl\mathbb{C}$ given by $\{\tilde{A}_-,\, \tilde{A}_0,\, \tilde{A}_+\}$, where*

$$\tilde{A}_- = \{z \in cl\mathbb{C} :\ Re\tilde{\rho}(z) < 0\},$$
$$\tilde{A}_0 = \{z \in cl\mathbb{C} :\ Re\tilde{\rho}(z) = 0\},$$
$$\tilde{A}_+ = \{z \in cl\mathbb{C} :\ Re\tilde{\rho}(z) > 0\}.$$

The next results has immediate proofs (see [6]).

Lemma 1.1. *Let $\tilde{f}$ a given function and $\tilde{R}$ an approximation. If we define*

$$f(z) = e^{\tilde{f}(z)}, \quad R(z) = e^{\tilde{R}(z)}, \quad \rho(z) = R(z)/f(z), \quad \tilde{\rho}(z) = \tilde{R}(z) - \tilde{f}(z).$$

The first kind order star of ρ is the second kind order star of $\tilde{\rho}$.

Theorem 1.3. *If z_o in $cl\mathbb{C}$ is a point in which $\tilde{\rho}$ is analytic and an interpolation of order p, then z_o belongs to A_0, $\iota(z_o) = p$ and it is a regular point.*

We say that a closed curve in $\tilde{A}_0$, positively oriented, is an A_+-loop if it is bounded by the inside of A_+, and we say that it is an A_--loop if it is bounded by the inside of A_-. Note that a closed curve may be not an A_+-loop nor an A_--loop and that the components of an A_+-loop may be an A_--loop too. We say that a loop has multiplicity L if it has exactly L interpolation points.

Theorem 1.4. *The multiplicity of an A_+-loop or an A_--loop is the number of singularities of $\tilde{\rho}$ in this loop. Furthermore, interpolation points and singularities points entwine himself along the loop.*

Theorem 1.5. *Let z_0 be a pole of $\tilde{\rho}$ with multiplicity q. Then $\iota(z_o) = q$ and it is a regular point.*

2. Linear Multistep Methods

Given an ordinary differential equation

$$y\prime = f(x, y), \; y(a) = \eta, \tag{2.1}$$

for which we assume to have a unique solution. If we denote by y_n an approximation of the solution in the point $x_n = a + nh$, $h > 0$ fix, $n = 0, 1, 2, \ldots$, and define $f_n \equiv f(x_n, y_n)$. We can represent a numerical linear multistep method [7] in its general form as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \quad \alpha_k = 1, \; |\alpha_0| + |\beta_0| \neq 0. \tag{2.2}$$

Clearly (2.2) may be characterized by the polynomials

$$r(z) = \sum_{j=0}^{k} \alpha_j z^j, \quad s(z) = \sum_{j=0}^{k} \beta_j z^j,$$

called *characteristics polynomials* of the method.
By definition [7] we say that the method (2.2) has order p when

$$\begin{aligned} \pounds[y(x_n) : h] &= \sum_{j=0}^{k} [\alpha_j y(x_{n+j}) - h\beta_j f(x_{n+j}, y(x_{n+j}))] \\ &= C_{p+1} h^{p+1} y^{(p+1)}(x_n) + O(h^{p+2}), \end{aligned} \tag{2.3}$$

for $y(x)$ sufficiently differentiable, where

$$C_0 = \sum_{j=0}^{k} \alpha_j, \qquad C_q = \frac{1}{q!} \sum_{j=0}^{k} j^q \alpha_j - \frac{1}{(q-1)!} \sum_{j=0}^{k} j^{q-1} \beta_j, \quad q \geq 1. \tag{2.4}$$

When we use a numerical method to solve an ordinary differential equation it is of fundamental interest to know that it has good properties of convergence, consistency and stability as the next definitions.

Definition 2.1 (Convergence [7])**.** *A linear multistep method (2.2) is called convergent when the following hold, to the solution $y(x)$ of the initial value problem (2.1),*

$$\lim_{h\to 0} y_n = y(x_n)$$

for all $x \in [a, b]$ and all solution $\{y_n\}$ of the difference equation (2.2) with initial condition $y_\mu = \eta_\mu(h)$ for which $\lim_{h\to 0} \eta_\mu(h) = \eta,\ \mu = 0, 1, \ldots, k-1$.

Definition 2.2 (Consistency and Zero-stability [7])**.** *We say that the method (2.2) is consistent when it has order $p \geq 1$ and we say that it is zero-stable if all roots of $r(z)$ has modulus less than or equals one and if that ones with modulus one is simple.*

Note that if (2.2) is consistent and zero-stable then follows from (2.4) that $r(1) = 0$ and $dr(1)/dt = s(1) \neq 0$.
We finish this section with a well-known result.

Theorem 2.1 (Dahlquist, 1956 [4])**.** *Linear multistep method (2.2) is convergent if, and only if, it is consistent and zero-stable.*

2.1. The First Dahlquist Barrier

In this section we will analyze the following theorem. To simplify our comments we will assume in this section that r and s has no common roots, since its roots has no influence in our analysis (see the proof of Theorem 2.3).

Theorem 2.2 (Dahlquist, 1956 [6])**.** *A convergent method given by (2.2) has order bounded by $2\lfloor (k+2)/2 \rfloor$ (two times the integer part of (k+2)/2). The order may be greater than $k+1$ just if k is even and all roots of r has modulus one. If $\beta_k \leq 0$ the order is bounded by k.*

To prove this theorem we need of some preliminary results as follows. By a question of space some of then has no proof (see [6]).

Theorem 2.3. *[4] A linear multistep method (2.2) has order p if, and only if,*

$$M(\log\, w,\, w) = r(w) - s(w)\, log\, w = O(|w-1|^{p+1}).$$

Proof: Suppose that the method has order p. Thus

$$\pounds[e^x : h] = e^x[r(e^h) - h\, s(e^h)] = e^x C_{p+1} h^{p+1} + O(|h|^{p+2}), \tag{2.5}$$

where $h \to 0$ and $C_{p+1} \neq 0$. But this is equivalent to say that $f(h) = M(h, e^h)$ which is analytic in $h = 0$ and has a zero of order $p+1$ in that point. As $w = e^h$ maps neighborhood of $h = 0$ onto neighborhood of $w = 1$ it follows (see [1, p.135]) that $f(\ln\, w) = M(\ln\, w,\, w)$ has a zero of order $p+1$ in $w = 1$, that is

$$M(\ln\, w,\, w) = O(|w-1|^{p+1}).$$

From the other side, if M has a zero of order p in $w = 1$ then from the same argument follows that equation (2.5) remain true. Since all constants α_j and β_j in (2.4) does not depends of the choice of $y(x)$ the proof follows. □

Note that we have $2k+1$ coefficients in the method (2.2) and hence we can expect to obtain a maximal order of $p = 2k$. Furthermore the order is p if, and only if,

$$r(w)/s(w) = \ln w + O(|w-1|^{p+1}).$$

In this way the maximal order occur just when $r(w)/s(w)$ is the Padé approximation (see [6]) to the logarithm in $w = 1$.
The maximal order method given by (2.2) has the following coefficients [6]

$$\alpha_j = \chi_k^{-1}(\chi_k - \chi_{k-j})\binom{k}{j}^2, \quad \beta_j = \frac{1}{2}\chi_k^{-1}\binom{k}{j}^2, \ j = 0, 1, 2, \ldots, k,$$

with

$$\chi_0 = 0, \ \chi_m = \sum_{j=1}^{m} \frac{1}{j}, \ m = 1, 2, \ldots.$$

Some calculation shows that $z^k r(z^{-1}) = -r(z)$. Thus, if $r(w) = 0$ then $r(w^{-1}) = 0$, that is, if some roots of r does not has modulus one then the zero-stability condition are not fulfilled.
Suppose that the method is zero-stable and that it has maximal order to some $k \geq 2$. In this case the sum of the modulus of the roots of r does not exceed the value of k. Meanwhile, this sum is given by $-\alpha_{k-1}$ and follows that $|\alpha_{k-1}| \leq k$. Using the explicit form of α_{k-1} we have

$$|\alpha_{k-1}| = \left| \frac{1}{\sum_{j=1}^{k-1}\frac{1}{j}} \left(\sum_{j=1}^{k-1}\frac{1}{j} - 1\right) \binom{k}{k-1}^2 \right| \Longrightarrow \frac{1}{\sum_{j=1}^{k-1}\frac{1}{j}} \left(\sum_{j=1}^{k-1}\frac{1}{j} - 1\right) k(k-1) \leq 0,$$

which is a contradiction when $k > 2$. In this way we conclude that a linear multistep method of k steps may not have maximal order and be zero-stable when $k > 2$.
We are now in a position in which the order star theory is welcome, since two conditions about polynomials, which are determined by geometrical properties of the complex plane , are in conflict.
The maximal order question attainable according with zero-stability is well-known as the first Dahlquist barrier and are already studied in 1956 with no help of the order stars theory. We will analyze this question here with a help of this theory. To this end we will use the second kind order star

$$\tilde{\rho}(z) = \frac{s(e^z)}{r(e^z)} - \frac{1}{z},$$

in virtue of Theorem (2.3). This 'inversion' maps the zeros of r onto the poles of $\tilde{\rho}$, they are thus points of non analyticity of the order star.

Figure (2) shows, respectively, the order stars of the Adams-Mouton method [4] of fifth order,

$$y_{i+4} - y_{i+3} = \frac{1}{720}h(251f_{i+4} + 646f_{i+3} - 264f_{i+2} + 106f_{i+1} - 19f_i),$$

and of the Milne's method [4] of fourth order,

$$y_{i+2} - y_i = \frac{1}{3}h(f_{i+2} + 4f_{i+1} + f_i).$$

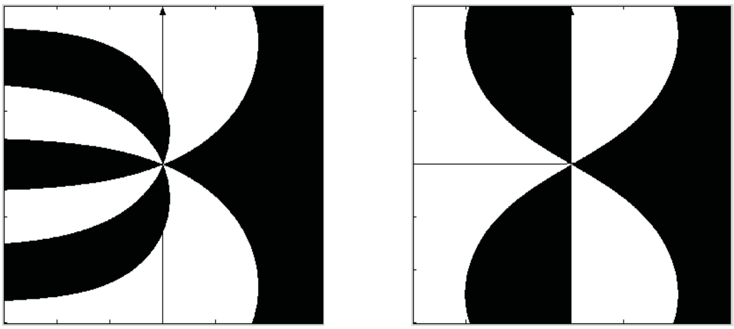

Figure 2: Order stars of second kind, $\tilde{A_+}$ in black.

Theorem 2.4. *The zero-stability conditions is equivalent to all poles of $\tilde{\rho}$ being in the complex semi-plane $Re z \le 0$ and that poles along $i\mathbb{R}$ being simples.*

Theorem 2.5. *If the method (2.2) has order $p \ge 2$ then $\iota(0) = p - 1$ and the origin is a regular point.*

Proof: It follows from Theorem (2.3) that

$$M(\ln z, z) = r(z) - s(z)\ln z = O(|z-1|^{p+1}).$$

The same arguments in that proof implies that

$$r(e^z) - z\, s(e^z) = O(|z|^{p+1}).$$

Hence

$$z\, s(e^z) = r(e^z) + O(|z|^{p+1}) \;\Rightarrow\; \frac{s(e^z)}{r(e^z)} = \frac{1}{z} + O(|z|^{p-1}),$$

since $r(1) = r(e^0) = 0$. Thus $z = 0$ is an interpolation point of order $p - 1$ of $f(z) = z^{-1}$, with $R(z) = s(e^z)/r(e^z)$. And the proof follows from Theorem (1.3).□

The function $\tilde{\rho}(z)$ involves values e^z which is periodic of period $2\pi i$ and this may complicate the counting of zeros and poles which is important here. At this form we will work in the set $S = \{z \in cl\mathbb{C} : |Im\, z| \le \pi\}$. Let S^o be the interior of S, $S^+ = S^o \cap \{Re\, z > 0\}$ and $S^- = S^o \cap \{Re\, z < 0\}$.

Lemma 2.1. *There exists $\kappa \in \mathbb{R}$ such that $\{z \in \mathbb{C} : Re\, z > \kappa\} \cap S$ is in A_+ or in A_-. If $\beta_k > 0$ then it be in A_+ otherwise it is in A_-.*

Note that poles and zeros of $\tilde{\rho}(z)$ are in A_0. This will be important in our analysis to determine its relative positions. Remember that a loop is a closed curve in A_0.

Lemma 2.2. *Let γ be a curve for which $\gamma \cap S \neq \varnothing$ and $\gamma \cap \partial S = \varnothing$. Then there is on γ exactly one pole of $\tilde{\rho}$ between any two roots of $\tilde{\rho}(z) = 0$. Furthermore, all poles of $\tilde{\rho}$ in S^o is a regular point with index equal its multiplicity.*

Let U be a bounded A_+-region or A_--region and suppose that $cl\mathbb{C} \cap \{\mathbb{R} + \pi i\} \neq \varnothing$. We will define $x_- = \min\{x \in \mathbb{R} : x + \pi i \in clU\} > -\infty$ and $x_+ = \max\{x \in \mathbb{R} : x + \pi i \in clU\} < \infty$.

Lemma 2.3. *Let $z_o \in \partial U \cap S^o$ be a zero of $\tilde{\rho}$. Then*
- If U is an A_+-region then or $x_+ + \pi i$ is a pole of $\tilde{\rho}$ or there exists a pole along the positive oriented curve in ∂U from z_o to $x_+ + \pi i$.
- If U is an A_--region then or $x_- + \pi i$ is a pole of $\tilde{\rho}$ or there exists a pole along the positive oriented curve in ∂U from $x_- + \pi i$ to z_o.
Furthermore, a similar condition holds if we works with a condition $\mathbb{R} - \pi i$ in the definition of U.

Let

$$F(t) = Re\{s(e^{it})r(e^{-it})\} = |r(e^{-it})|^2 Re\, \tilde{\rho}(it),\ t \in \mathbb{R}.$$

Some calculations shows that F is a polynomial of degree k in $(1 - \cos t)$. Thus it is an even function. Furthermore, from Theorem 2.3 follows that

$$r(e^{-it}) = -its(e^{-it}) + O(|t|^{p+1}).$$

And we can see that

$$F(t) = O(|t|^{2\lfloor(p+2)/2\rfloor}) = O((1 - \cos t)^{\lfloor(p+2)/2\rfloor})$$

and that there are exactly two cases to consider: or $F \equiv 0$ or there are at most $k - \lfloor(p + 2)/2\rfloor$ zeros in $(0, \pi)$.

We will suppose that the method (2.2) is zero-stable to first analyze the cases $\beta_k > 0$. If $F \equiv 0$ then $i\mathbb{R} \in \tilde{A}_0$ and follows from Theorem 2.5 that exactly $p - 1$ sectors of $\tilde{A}_+$ and $\tilde{A}_-$ are close to the origin from the inside of S_+ and that it is regular (see Figure 2). Since the imaginary axis is in $\tilde{A}_0$ no loop can cross S_-. Furthermore, from Lemma 2.1 there exists just a region $\tilde{A}_+$ unbounded in S_+. Hence, may exists at most $p - 2$ loops in S_+. By lemmas 2.2 and 2.3 each loop has a pole of $\tilde{\rho}$. Therefore, by zero-stability, does not exists poles in S_+ and that poles $i\mathbb{R}$ are simples. It follows then that each loop has part of the imaginary axis. From Theorem 1.5 and from the fact that the poles are simple in $i\mathbb{R}$ we have that each pole is in just one loop. As $r(e^0) = r(1) = 0$ and $z = 0$ are not poles we conclude that there are at most $k - 1$ poles which may be in some loop. In this way $p - 2 \leq k - 1$. Therefore, if k is even we can count two times the same pole two if it is $\pm\pi i$. Thus $p - 2 \leq k$ and we have $p \leq 2\lfloor(k + 2)/2\rfloor$.

A similar argument applied to the other cases implies the given bounds and this bounds produces the first Dahlquist barrier. □

3. Final Remarks

Order stars theory is very important in the study of properties of numerical methods and produces simple and elegant proofs. It is also important because it gives consistence to numerical analysis by making a link between it and the complex variable function theory.

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Resumo. A teoria de Order Stars introduzida por Wanner tornou-se uma ferramenta fundamental para o estudo e a compreensão das propriedades de métodos numéricos, tais como a ordem e a estabilidade. Neste trabalho procuramos estudar algumas propriedades dos métodos numéricos lineares de passos múltiplos utilizando esta teoria.

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