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Equations of the Elastic of a Spatial Lattice from a Continuous Model

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ABSTRACT. Within metal constructions there are structures made up of parallel members called legs, connected to each other by diagonals. Some of these structures, with a triangular cross section, turn out to be the guyed masts used in the communications industry, but also columns and beams used in metal structures. In the present work, the equations that allow obtaining the equations of the elastic of a spatial lattice of triangular cross section are developed, in which the legs are joined together by zig-zag diagonals. To do this, we start from an energy proposal in which the differential equations of equilibrium and the boundary conditions of the proposed problem are determined. Finally, examples are presented where results are obtained with the equations developed, and they are compared with the results obtained from the application of the expressions given in CIRSOC 303 Recommendation (1996) and from finite element models.

Keywords: spatial lattice, static, strain energy, differential equations.

1 INTRODUCTION

Within metallic constructions there are various structures made up of parallel members called legs, connected to each other by diagonals. Some of these structures, with a triangular cross section, turn out to be the guyed masts used in the communications industry, but this typology is also frequently used to constitute columns and beams in various metallic structures.

These structures have a significant number of elements (legs and diagonals), so it is common in the design to use equivalent models that arise from replacing the lattice with an equivalent beam-column type formulation, resulting in a lower cost at the time of the analysis [8,9]. For the determination of this formulation, there are approaches based on the analysis of flexibility [3,7], as well as others based on energy issues such as the one carried out by Guzmán *et al.* [5]. In this last approach, the differential equations and the boundary conditions that govern the mechanical

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behavior of a spatial lattice of triangular cross section were obtained, with diagonals arranged in a zig-zag pattern. But this approach has also allowed the development of another alternative form of representation of lattices, both triangular and rectangular cross-section, from obtaining the equivalent properties necessary for modeling the problem as beam-column [4, 6].

In the present work, and making use of what was developed in [5], the equations are obtained that allow to directly determine the equations of the elastic of the analyzed lattice. Finally, examples are presented where results are obtained with the proposed equations, and they are compared with the results obtained from the application of the expressions given in the CIRSOC 303 recommendation [2] and from finite element models, thus allowing conclusions to be drawn.

2 SPATIAL LATTICE ANALYZED

The analyzed spatial lattice of total height L (Figure 1), presents a triangular cross section of sides e and constituted by three continuous legs joined together by three planes of equal diagonals and articulated at their ends, following a zig-zag pattern.

The section of each leg is A_l and that of each diagonal is A_d. The moment of inertia of the legs with respect to each of the main directions is J_{ly} and J_{lz}, while the length of each diagonal is L_d. Δ is the separation or step between them. The material that is part of each element is elastic and linear, with a modulus of elasticity E.

Starting from the energetic approach, the sum of the energy developed in each of the elements that make up the lattice is initially determined, and then accepting the hypothesis that the diagonals are sufficiently numerous, the summations are approximated by classical Cauchy-Riemann integrals thus allowing the passage from the discrete domain to the continuous domain.

In this way, energy functionals are available that allow obtaining the Lagrangian of the system, and by applying the Hamilton principle to said functional, obtaining the differential equations and the edge conditions that govern the mechanical behavior of the analyzed lattice, thus giving rise to to a continuous model of representation.

These differential equations constitute a linear system of nine equations in terms of the spatial x and temporal variables t. In reference [5] the development of the afore mentioned approach is presented. Next, the differential system and and boundary conditions that governs the behavior of the continuous model is rewritten, and that for the present case of the elastic model, only considers the spatial variable x.

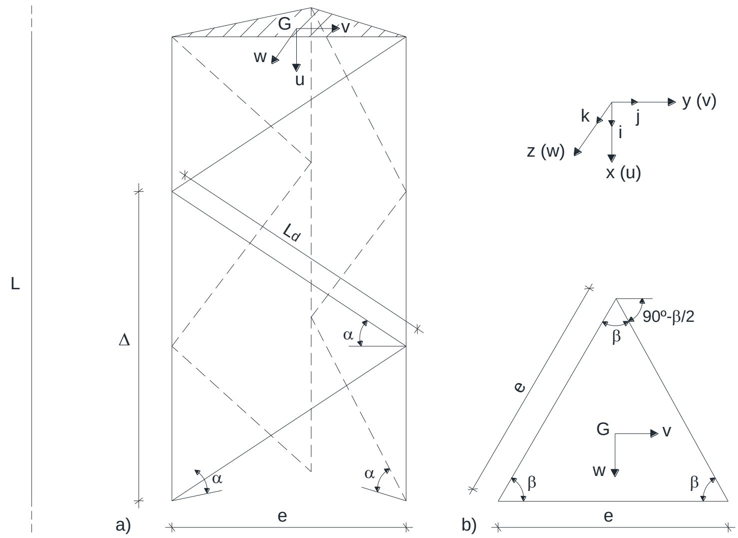

Figure 1: Lattice analyzed. a) Elevation. b) Cross section.

$$
\begin{aligned}
E\left[u_a''(x)+K_u\left(\delta_1(x)-\delta_3(x)+\tfrac{\Delta}{2}\eta_3'(x)\right)\right]&=0\\
E\left[u_b''(x)+K_u\left(\delta_2(x)-\delta_1(x)+\tfrac{\Delta}{2}\eta_1'(x)\right)\right]&=0\\
E\left[u_c''(x)+K_u\left(\delta_3(x)-\delta_2(x)+\tfrac{\Delta}{2}\eta_2'(x)\right)\right]&=0\\
E\left[v_a''''(x)+K_v\left(\eta_1(x)+\tfrac{\eta_3(x)}{2}-\Delta\delta_3'(x)\right)\right]&=0\\
E\left[v_b''''(x)-K_v\left(\tfrac{\eta_2(x)}{2}+\eta_1(x)-\tfrac{\Delta}{2}\delta_1'(x)\right)\right]&=0\\
E\left[v_c''''(x)-K_v\left(\tfrac{\eta_3(x)}{2}-\tfrac{\eta_2(x)}{2}+\Delta\delta_2'(x)\right)\right]&=0\\
E\left[w_a''''(x)+K_w\sqrt{3}\left(\tfrac{\eta_3(x)}{2}-\Delta\delta_3'(x)\right)\right]&=0\\
E\left(w_b''''(x)+K_w\tfrac{\sqrt{3}}{2}\eta_2(x)\right)&=0\\
E\left[w_c''''(x)-K_w\tfrac{\sqrt{3}}{2}\left(\eta_2(x)+\eta_3(x)-2\Delta\delta_2'(x)\right)\right]&=0
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&\left|\left[E\left(u_a'(x)+K_u\eta_3(x)\tfrac{\Delta}{2}\right)\right]U_a(x)\right|_{x=0}^{x=L}=0\\
&\left|\left[E\left(u_b'(x)+K_u\eta_1(x)\tfrac{\Delta}{2}\right)\right]U_b(x)\right|_{x=0}^{x=L}=0\\
&\left|\left[E\left(u_c'(x)+K_u\eta_2(x)\tfrac{\Delta}{2}\right)\right]U_c(x)\right|_{x=0}^{x=L}=0\\
&\left|\left[E\left(v_a'''(x)-K_v\delta_3(x)\tfrac{\Delta}{4}\right)\right]V_a(x)\right|_{x=0}^{x=L}=0\\
&\left|\left[E\left(v_b'''(x)+K_v\delta_1(x)\tfrac{\Delta}{4}\right)\right]V_b(x)\right|_{x=0}^{x=L}=0\\
&\left|\left[E\left(v_c'''(x)-K_v\delta_2(x)\tfrac{\Delta}{4}\right)\right]V_c(x)\right|_{x=0}^{x=L}=0\\
&\left|\left[E\left(w_a'''(x)-K_w\delta_3(x)\sqrt{3}\tfrac{\Delta}{4}\right)\right]W_a(x)\right|_{x=0}^{x=L}=0\\
&\left|E\left(w_b'''(x)\right)W_b(x)\right|_{x=0}^{x=L}=0\\
&\left|\left[E\left(w_c'''(x)+K_w\delta_2(x)\sqrt{3}\tfrac{\Delta}{4}\right)\right]W_c(x)\right|_{x=0}^{x=L}=0\\
&\left|EJ_{ly}\left(v_a''(x)\right)V_a'(x)\right|_{x=0}^{x=L}=0\,;\ \left|EJ_{ly}\left(v_b''(x)\right)V_b'(x)\right|_{x=0}^{x=L}=0\\
&\left|EJ_{ly}\left(v_c''(x)\right)V_c'(x)\right|_{x=0}^{x=L}=0\,;\ \left|EJ_{lz}\left(w_a''(x)\right)W_a'(x)\right|_{x=0}^{x=L}=0\\
&\left|EJ_{lz}\left(w_b''(x)\right)W_b'(x)\right|_{x=0}^{x=L}=0\,;\ \left|EJ_{lz}\left(w_c''(x)\right)W_c'(x)\right|_{x=0}^{x=L}=0
\end{aligned}
\tag{2.2}
$$

The coefficients K_i, η_m and δ_m, with i = u,v,w and with m =1,2,3 represents:

$$K_u = \frac{A_d}{2A_l L_d^2}$$
$$K_v = \frac{A_d e}{J_{ly}\Delta L_d^2} \tag{2.3}$$
$$K_w = \frac{A_d e}{J_{lz}\Delta L_d^2}$$

$$\eta_1(x) = \frac{\Delta^2}{2L_d}u_b'(x) + \frac{2e}{L_d}\Big(v_a(x) - v_b(x)\Big)$$
$$\eta_2(x) = \frac{\Delta^2}{2L_d}u_c'(x) - \frac{e}{L_d}\Big(v_b(x) - v_c(x)\Big) + \frac{\sqrt{3}e}{L_d}\Big(w_b(x) - w_c(x)\Big) \tag{2.4}$$
$$\eta_3(x) = \frac{\Delta^2}{2L_d}u_a'(x) - \frac{e}{L_d}\Big(v_c(x) - v_a(x)\Big) - \frac{\sqrt{3}e}{L_d}\Big(w_c(x) - w_a(x)\Big)$$

$$\delta_1(x) = \frac{\Delta}{L_d}\Big(u_b(x) - u_a(x)\Big) - \frac{\Delta e}{L_d}v_b'(x)$$
$$\delta_2(x) = \frac{\Delta}{L_d}\Big(u_c(x) - u_b(x)\Big) + \frac{\Delta e}{2L_d}v_c'(x) - \frac{\sqrt{3}\Delta e}{2L_d}w_c'(x) \tag{2.5}$$
$$\delta_3(x) = \frac{\Delta}{L_d}\Big(u_a(x) - u_c(x)\Big) + \frac{\Delta e}{2L_d}v_a'(x) + \frac{\sqrt{3}\Delta e}{2L_d}w_a'(x)$$

3 ELASTIC EQUATIONS

From the continuous representation model indicated above, the equations are obtained that allow determining, in this case, the elastic of a simply supported lattice subject to a uniformly distributed load ($q_z/3$) on each beam, and applied in the direction of w (Figure 2).

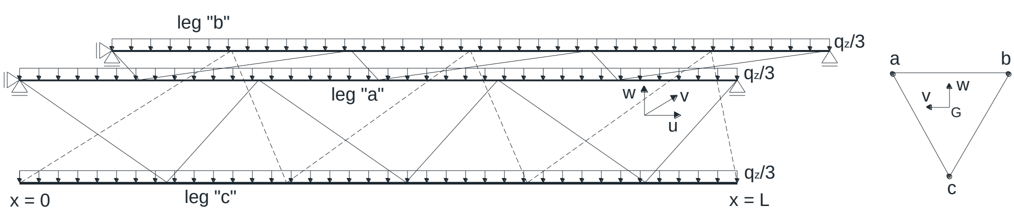

Figure 2: Spatial lattice simply supported and subjected to a distributed load.

It should be noted that in the analysis of the elastic of composite parts, the local inertia of each part (J_l) and in each main direction is significantly lower than the global inertia of the section (J). This situation, which is very frequent in practice, makes it possible to set aside in the differential system the contribution of $v_i''''(x)$ and $w_i''''(x)$ (i = a,b,c), terms that energetically they come from the bending of each leg of the lattice. In this case the global moment of inertia of the section according to Steiner, results:

$$
\begin{aligned}
J_y &= 3J_{ly} + 2A_l\left(\tfrac{e}{2}\right)^2 \\
J_z &= 3J_{lz} + 2A_l\left(\tfrac{e}{2}\right)^2
\end{aligned}
\tag{3.1}
$$

Assuming the simplification indicated above, the global inertias are rewritten as:

$$
J_y = J_z = A_l \tfrac{e^2}{2} \tag{3.2}
$$

The natural and kinematic boundary conditions of the analyzed example result:

$$
\begin{aligned}
&E\left(u_a'(0) + K_u\eta_3(0)\tfrac{\Delta}{2}\right) = E\left(u_a'(L) + K_u\eta_3(L)\tfrac{\Delta}{2}\right) = 0 \\
&E\left(u_b'(0) + K_u\eta_1(0)\tfrac{\Delta}{2}\right) = E\left(u_b'(L) + K_u\eta_1(L)\tfrac{\Delta}{2}\right) = 0 \\
&E\left(u_c'(0) + K_u\eta_2(0)\tfrac{\Delta}{2}\right) = E\left(u_c'(L) + K_u\eta_2(L)\tfrac{\Delta}{2}\right) = 0 \\
&u_a(0) = u_b(0) = w_a(0) = w_b(0) = w_a(L) = w_b(L) = 0
\end{aligned}
\tag{3.3}
$$

Therefore, solving the system of governing differential equations, we obtain the elasticity of the problem in the direction of the load, deflection (f) and maximum sectional rotation (θ):

$$
\begin{aligned}
f_{(L/2)} &= -\tfrac{5}{384}\tfrac{q_zL^4}{EJ_z} - \tfrac{1}{6}\tfrac{q_zL_d^3L^2}{EA_de^2\Delta} \\
\theta_{(L)} &= \tfrac{1}{24}\tfrac{q_zL^3}{EJ_z} + \tfrac{2}{3}\tfrac{q_zL_d^3L}{EA_de^2\Delta}
\end{aligned}
\tag{3.4}
$$

Now defining φ as the parameter that takes into account the shear deformation:

$$
\varphi = \tfrac{12A_lL_d^3}{A_d\Delta L^2} \tag{3.5}
$$

deflection and rotation can be rewritten as:

$$
\begin{aligned}
f_{(L/2)} &= -\tfrac{5}{384}\tfrac{q_zL^4}{EJ_z}\left(1 + \tfrac{8}{15}\varphi\right) \\
\theta_{(L)} &= \tfrac{1}{24}\tfrac{q_zL^3}{EJ_z}\left(1 + \tfrac{2}{3}\varphi\right)
\end{aligned}
\tag{3.6}
$$

When comparing these results with those given by the classical beam theory (Euler-Bernoulli) [1], substantial differences are observed. The latter is due to the fact that this theory does not consider shear deformations.

In the hypothetical case in which $A_d \to \infty$, it will turn out that $\varphi \to 0$, with which the expressions obtained coincide with those of the classical theory.

Regarding the expression given by the CIRSOC 303 Recommendation (adjusted to the nomenclature of this work) and for the determination of the deflection in the direction of w, it results:

$$f_{(L/2)CIRSOC} = -\frac{5}{384}\frac{q_z L^4}{EJ_z}\left(1+\frac{9.6}{16}\varphi\right) \tag{3.7}$$

When comparing equations 9 and 11, a difference between them is observed in what has been defined as the deflection factor δ, that is, in the term:

$$\delta = (1+K\varphi) \tag{3.8}$$

where K represents the constant that depends on the boundary and load conditions.

This difference, indicated for the case studied, varies approximately in the order of 1 to 4% as the shear factor φ increases from 0.1 to 1.0.

4 RESULTS

Figure 3 shows a comparison of the maximum deflection factors (δ) obtained with the expression given by the CIRSOC 303 Recommendation (C), with the developed equation (E), and with the results obtained from a finite element model (EF).

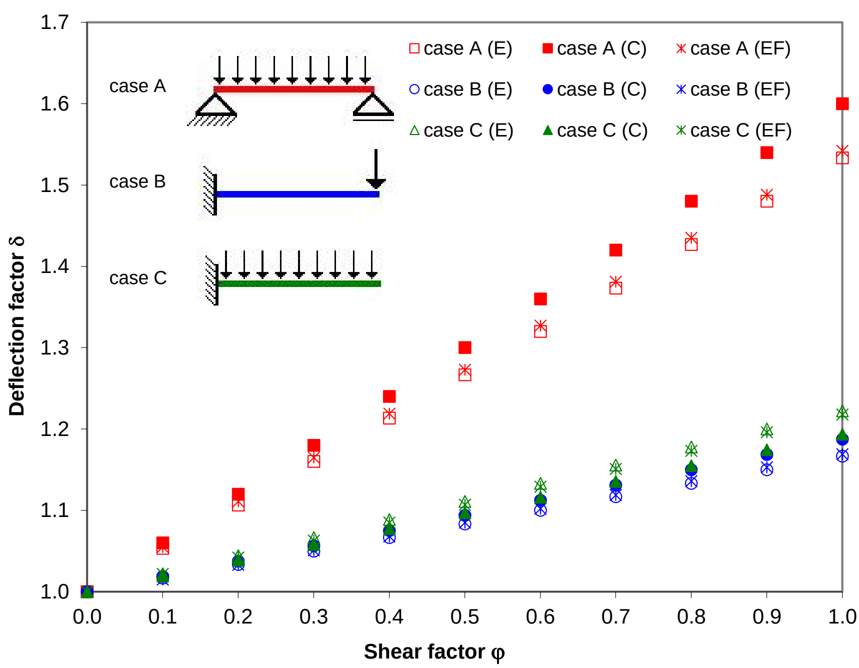

Figure 3: Comparison of deflection factor results obtained for the cases analyzed.

These results shown correspond to the cases of: i) reticulated simply supported with distributed load (case A); ii) cantilever lattice with point load (case B); iii) cantilever lattice with distributed load (case C). In the cases analyzed, the deflection factor δ was obtained for different values of the shear factor φ.

With a development similar to that indicated above, Table 1 presents the formulations obtained from the continuous model and the expressions given in CIRSOC 303, for the three cases analyzed (A, B and C), and the proposed equation for a fourth case corresponding to a fixed-fixed lattice with distributed load (case D). It should be noted that for the latter case, the CIRSOC Recommendation 303 does not present any expression.

Table 1: Expressions for the determination of deflection in the cases analyzed.

Case	BC	Load	Proposed equation	Expression CIRSOC 303
A	simply supported	uniformly distributed (q)	$\frac{5}{384}\frac{q_zL^4}{EJ_z}\left(1+\frac{8}{15}\varphi\right)$	$\frac{5}{384}\frac{q_zL^4}{EJ_z}\left(1+\frac{9.6}{16}\varphi\right)$
B	cantilever	Concentrated at the free end (P)	$\frac{1}{3}\frac{PL^3}{EJ_z}\left(1+\frac{1}{6}\varphi\right)$	$\frac{1}{3}\frac{PL^3}{EJ_z}\left(1+\frac{3}{16}\varphi\right)$
C	cantilever	uniformly distributed (q)	$\frac{1}{8}\frac{q_zL^4}{EJ_z}\left(1+\frac{2}{9}\varphi\right)$	$\frac{1}{8}\frac{q_zL^4}{EJ_z}\left(1+\frac{3.1}{16}\varphi\right)$
D	fixed-fixed	uniformly distributed (q)	$\frac{1}{384}\frac{q_zL^4}{EJ_z}\left(1+\frac{8}{3}\varphi\right)$	-

5 CONCLUSIONS

In the present work, the equations were obtained to determine the elastic of a spatial lattice of triangular cross section, with diagonals arranged in a zig-zag pattern. Said equations were obtained from a continuous representation model, where shear deformations are incorporated into the formulation. The results obtained when applying the proposed equations coincide with those obtained in the finite element models, but present differences with the classical beam theory since the latter does not consider shear deformation. On the other hand, when comparing the results with those obtained from applying the expressions given by the CIRSOC 303, they turned out to be significantly lower, which raises the possibility of proposing adjustments to the constant that accompanies the shear factor in those expressions.

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REFERENCES

[1] O. Belluzzi. "Ciencia de la construcción I". Aguilera, España (1973).

[2] CIRSOC-303. "Estructuras livianas de acero". INTI, Argentina (1996).

[3] C. Filipich & E. Bambill. Frecuencias naturales de piezas empresilladas vía serie de potencias. *Mecánica Computacional*, **22** (2003).

[4] A. Guzmán & V. Roldan. Equivalent properties for analysis as beam-column of spatial lattices of rectangular cross-section. *Advanced Steel Construction*, **17** (2021), 95–103.

[5] A. Guzmán, M. Rosales & C. Filipich. Natural vibrations and buckling of a spatial lattice structure using a continuous model derived from an energy approach. *International Journal of Steel Structures*, **17** (2017), 1–14.

[6] A. Guzmán, M. Rosales & C. Filipich. Continuous one-dimensional model of a spatial lattice. Deformation, vibration and buckling problems. *Engineering Structures*, **182** (2019), 290–300.

[7] N.B. Kahla. Equivalent beam-column analysis of guyed towers. *Computers and Structures*, **55** (1995), 631–645.

[8] M. Madagula, Y. Wahba & G. Monforton. Dynamic response of guyed mast. *Engineering Structures*, **20** (1998), 1097–1101.

[9] Y. Wahba, M. Madagula & G. Monforton. Evaluation of non-linear analisis of guyed antenna towers. *Computers and Structures*, **68** (1998), 207–212.