Trends in Computational and Applied Mathematics

Trends in Computational and Applied Mathematics, **23**, N. 3 (2022), 539-547
Sociedade Brasileira de Matemática Aplicada e Computacional
Online version ISSN 2676-0029
www.scielo.br/tcam
doi: 10.5540/tcam.2022.023.03.00539

A Note on the Well-Posedness of Control Complex Ginzburg-Landau Equations in Zhidkov Spaces

A. BESTEIRO

Received on August 26, 2021 / Accepted on March 24, 2022

ABSTRACT. In this note, we consider the Complex Ginzburg-Landau equations with a bilinear control term in the real line. We prove well-posedness results concerned with the initial value problem for these equations in Zhidkov spaces using splitting methods.

Keywords: well-posedness, Zhidkov spaces, Lie-Trotter method.

1 INTRODUCTION

In this note, we deal with the 1-dimensional system

$$\begin{cases} \partial_t u = (\alpha + i\beta)\partial_{xx} u + \gamma u + (c + id)|u|^2 u + (a + ib)v(x,t)u, \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where $u(x,t)$ is a complex valued function with $x \in \mathbb{R}$, $t > 0$, $\alpha > 0, \beta > 0, \gamma \geq 0$, $a,b,c,d > 0$ and v is a bounded control function. The linear term represented by $(\alpha + i\beta)\partial_{xx}$ characterizes the Complex Ginzburg-Landau equation (CGLe). For $\beta = 0$ (1.1) reduces to a nonlinear heat equation and for $\alpha = 0$ to a nonlinear Schrödinger equation. The cubic CGLe is one of the most important nonlinear equations with applications in physics. It describes a large number of linear and nonlinear phenomena from superconductivity, superfluidity and Bose-Einstein condensation to liquid crystals [1]. Well-posedness of (1.1) has been studied with different nonlinearities and in different spaces (see for instance, [4, 11, 12]). Our aim is to study the well-posedness of the Complex Ginzburg-Landau equation with a bilinear control term, in Zhidkov spaces, using splitting methods. Controllability problems in parabolic equations were studied with different control alternatives and nonlinearities [2, 3, 15, 16]. Zhidkov spaces were introduced by P. Zhidkov in [17] defined as bounded and uniformly continuous functions, with derivatives up to k order in L^2. Many applications were found for these spaces, for instance, in nonlinear optics, Zhidkov functions are used to model dark solitons. In [8], dark soliton solutions are described for a special case

Universidad Abierta Interamericana, Centro de Altos Estudios en Tecnología Informatica, Buenos Aires, Argentina – E-mail: agustin.besteiro@uai.edu.ar https://orcid.org/0000-0001-5243-6090

of the complex Ginzburg-Landau equation. A typical example of a function in Zhidkov spaces is described in [10, 13]. These are solutions of the form $u(x,t) = u_v(x - vt)$, in particular for the one dimensional case we have:

$$u_v(x) = \sqrt{1 - \frac{v^2}{2}} \tanh\left(\sqrt{1 - \frac{v^2}{2}} \frac{x}{\sqrt{2}}\right) + i\frac{v}{\sqrt{2}}$$

The goal of this article is to prove well-posedness of (1.1) for Zhidkov spaces in the real line, using splitting the result in [4]. These are numerical methods that split the flow of the equation, to approximate solutions. The time interval is divided into equal parts, and in each one, the equation evolves alternating the linear and nonlinear flows. This establishes an advantage for us, it exchanges a complicated problem (1.1), for two simpler equations. In this case, we extend the method to a "triple splitting", dividing the equation into three parts. It is important to remark that this same method can be applied to prove well-posedness for other well known equations such as, reaction-diffusion and Schrödinger control equations in L^p spaces. The splitting method is based on a Lie-Trotter method developed recently for numerical purposes [6, 14].

The paper is organized as follows: In Section 2 we set notations and state some preliminary results. In section 3 we analyze the nonlinear problem. Finally, in section 4 and using splitting methods, we combine results from sections 2 and 3 to show that the solution of (1.1) is in a Zhidkov space.

2 NOTATIONS AND PRELIMINARIES.

We introduce some definitions and preliminary results.

Definition 2.1. *We define $C_u(\mathbb{R})$ as the set of uniformly continuous and bounded functions on $\mathbb{R}$.*

Definition 2.2. *For $k > d/2$, we define the Zhidkov space as,*

$$X^k(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) \cap C_u(\mathbb{R}^d) : \ \partial_j u \in L^2(\mathbb{R}^d), 1 \leq |j| \leq k\}$$

equipped with the norm:

$$\|u\|_{X^k} = \|u\|_{L^\infty} + \sum_{1\leq|j|\leq k} \left\|\partial_j u\right\|_{L^2} \tag{2.1}$$

Remark 2.1. *Zhidkov spaces are closed for the norm defined in* (2.1). *(See [10])*

The following definitions and proofs can be extended to $x \in \mathbb{R}^d$ (See [9]).

Definition 2.3. *We denote $U(t)$ as the one parameter semigroup that solves the underlying linear equation*

$$\partial_t u = (\alpha + i\beta)\partial_{xx} u + \gamma u \tag{2.2}$$

The operator can be represented by the convolution in x

$$U(t)u_0 = (4\pi t(\alpha+i\beta))^{-1/2}e^{(-x^2/[4t(\alpha+i\beta)])+\gamma t} * u_0 = G_t(x) * u_0$$

and the kernel G_t satisfies:

$$|G_t(x)| = (4\pi t(\alpha^2+\beta^2)^{1/2})^{-1/2}e^{(-\alpha x^2/[4t(\alpha^2+\beta^2)])+\gamma t}$$

Clearly, $G_t(x) \in L^1(\mathbb{R})$.

Proposition 2.1. *The one-parameter family $\{U(t)\}_{t\geq 0}$ of operators defined as $U(t)u_0 = G_t * u_0$ is a strongly continuous semigroup on $C_u(\mathbb{R})$.*

Proof. The proof is similar to Proposition 2.2 in [5]. □

Lemma 2.1. *If $u_0 \in X^1(\mathbb{R})$ then $U(t)u_0 \in C([0,T^*(u_0)),X^1(\mathbb{R}))$ for $t > 0$*

Proof. As $u_0 \in L^\infty(\mathbb{R})$ and $G_t(x) \in L^1(\mathbb{R})$ then using Young's inequality we have

$$\|G_t * u_0\|_{L^\infty} \leq \|G_t\|_{L^1}\|u_0\|_{L^\infty}$$

On the other hand, we obtain

$$\|\partial_x(G_t * u_0)\|_{L^2} = \|G_t * \partial_x u_0\|_{L^2} \leq \|G_t\|_{L^1} * \|\partial_x u_0\|_{L^2}$$

As $G_t \in L^1(\mathbb{R})$ and $\partial_x u_0 \in L^2(\mathbb{R})$ we have the result. □

Remark 2.2. *Similarly, if $x \in \mathbb{R}^d$ and we have k derivatives of $U(t)u_0$, a similar procedure proves that $U(t)u_0 \in C([0,T^*(u_0)),X^k(\mathbb{R}^d))$.*

Next, we consider integral solutions of the problem (1.1). We say that $u \in C([0,T],C_u(\mathbb{R}))$ is a mild solution of (1.1) if and only if u verifies

$$u(t) = U(t)u_0 + \int_0^t U(t-t')B(x,t',u(t'))dt'. \tag{2.3}$$

where $B(x,t,u) = (c+id)|u|^2u + (a+ib)v(x,t)u$. If B is a locally Lipschitz map, for any $z_0 \in C_u(\mathbb{R})$ there exists a unique solution of the equation

$$\begin{cases} \partial_t z = B(t,z), \\ z(0) = z_0, \end{cases} \tag{2.4}$$

defined in the interval $[0,T^*(z_0))$. Moreover, there exists a nonincreasing function $\bar{T} : [0,\infty) \to [0,\infty)$, such that $T^*(z_0) \geq \bar{T}(|z_0|)$. The solution of (2.4) is solution of the integral equation

$$z(t) = z_0 + \int_0^t B(t',z(t'))dt'. \tag{2.5}$$

Also, one of the following alternatives holds:

- $T^*(z_0) = \infty$;
- $T^*(z_0) < \infty$ and $|z(t)| \to \infty$ when $t \uparrow T^*(z_0)$.

We denote by $N : \mathbb{R}^+ \times \mathbb{R}^+ \times C_u(\mathbb{R}) \to C_u(\mathbb{R})$ the flow generated by the ordinary equation, i.e.: for any $x \in \mathbb{R}$, $N(t,t_0,u_0)(x)$ is the solution of the problem (2.4) with initial datum $z_0 = u_0(x)$. Therefore, if $u(t) = N(t,t_0,u_0)$

$$u(x,t) = u_0(x) + \int_0^t B(x,t',u(x,t'))dt'$$

We recall a well-known local existence result for evolution equations.

Theorem 2.1. *There exists a function $T^* : C_u(\mathbb{R}) \to \mathbb{R}_+$ such that for $u_0 \in C_u(\mathbb{R})$, exists a unique $u \in C([0,T^*(u_0)),C_u(\mathbb{R}))$ mild solution of* (1.1) *with $u(0) = u_0$. Moreover, one of the following alternatives holds:*

- $T^*(u_0) = \infty$*;*
- $T^*(u_0) < \infty$ *and* $\lim_{t\uparrow T^*(u_0)} \sup_{x\in\mathbb{R}} |u(t)| = \infty$.

Proof. See Theorem 4.3.4 in [7]. □

Proposition 2.2. *Under conditions of the theorem above, the following statements hold true:*

1. $T^* : C_u(\mathbb{R}) \to \mathbb{R}_+$ *is lower semi-continuous;*
2. *If $u_{0,n} \to u_0$ in $C_u(\mathbb{R})$ and $0 < T < T^*(u_0)$, then $u_n \to u$ in the Banach space $C([0,T],C_u(\mathbb{R}))$.*

Proof. See Proposition 4.3.7 in [7]. □

3 NONLINEAR EQUATION

In this section, we first analyze the following control equation:

$$\begin{cases} \partial_t u = (a+ib)v(x,t)u, \\ u(0) = u_0 \end{cases} \tag{3.1}$$

where $v \in C([0,T^*(u_0)),L^\infty(\Omega))$ and Ω is a bounded interval of $\mathbb{R}$ with $supp(v(\cdot,t)) \subseteq \Omega$. The following Lemma allows us to have well-posedness of the control equation in $X^1(\mathbb{R})$ which is essential to apply the splitting method.

Lemma 3.2. *Let $v \in C([0,T^*(u_0)),L^\infty(\Omega))$ with $\partial_x v \in C([0,T^*(u_0)),L^\infty(\Omega))$ and let $B(x,t,u) = (a+ib)v(x,t)u$, then $B : \mathbb{R}^+ \times \mathbb{R}^+ \times X^1(\mathbb{R}) \to X^1(\mathbb{R})$ is a well-defined operator and $B(x,t,u)$ is a locally Lipschitz map in u.*

Proof. Let $u \in X^1(\mathbb{R})$ then

$$\begin{aligned}\|B(x,t,u)\|_{X^1} &= \|(a+ib)v(x,t)u\|_{X^1} \leq K(\|v(x,t)u\|_{L^\infty} + \|\partial_x(v(x,t)u)\|_{L^2})\\ &= K(\|v(x,t)u\|_{L^\infty} + \|\partial_x v(x,t)u + v(x,t)\partial_x u\|_{L^2})\\ &\leq K(\|v(x,t)u\|_{L^\infty} + \|\partial_x v(x,t)u\|_{L^2} + \|v(x,t)\partial_x u\|_{L^2})\end{aligned}$$

As $v(x,t)$ is a bounded function in Ω then $\|B(x,t,u)\|_{X^1} < \infty$. On the other hand, using the notation $B(x,t,u) = B(u)$, we have

$$\begin{aligned}\|B(u)-B(w)\|_{X^1} &= \|(a+ib)v(x,t)u-(a+ib)v(x,t)w\|_{X^1} = \|(a+ib)v(x,t)(u-w)\|_{X^1}\\ &\leq K\|v(x,t)(u-w)\|_{X^1} \leq K\|v(x,t)(u-w)\|_{L^\infty} + K\|\partial_x(v(x,t)(u-w))\|_{L^2}\\ &\leq K\|v(x,t)(u-w)\|_{L^\infty} + K\|(\partial_x v(x,t))(u-w)\|_{L^2} + K\|(v(x,t))(\partial_x(u-w))\|_{L^2}\\ &\leq K\|v(x,t)\|_{L^\infty}\|(u-w)\|_{L^\infty} + K\|(\partial_x v(x,t))\|_{L^\infty}\|(u-w))\|_{L^2}\\ &+ K\|(v(x,t))\|_{L^\infty}\|(\partial_x(u-w))\|_{L^2}\\ &= K'\|(u-w)\|_{L^\infty} + K''\|(u-w))\|_{L^2} + K'\|(\partial_x(u-w))\|_{L^2} \leq k\|u-w\|_{X^1}\end{aligned}$$

where in the last step we used Hölder's inequality. □

We have the same result for the solution for the nonlinear problem associated with the term $|u|^2u$, that is the equation

$$\begin{cases}\partial_t u = -(c+id)|u|^2u,\\ u(0) = u_0,\end{cases} \tag{3.2}$$

Lemma 3.3. *If $u_0 \in X^1(\mathbb{R})$ then the solution of the problem* (3.2)*, $u(t) \in C([0,T^*(u_0)),X^1(\mathbb{R}))$.*

Proof. See [4] Lemma 3.1. □

4 SPLITTING METHOD

This section is based on the splitting method developed in [5, 6, 14]. We apply the Lie-Trotter method to the linear and nonlinear problem. The temporal variable must be broken down into regular intervals and the evolution of the linear, nonlinear and control problems are considered alternately. This is described by three sequences: $\{V_{0,k}\}$ for the linear equation and $\{V_{1,k}\},\{W_{2,k}\}$ for the nonlinearity and the control term, respectively. Using Theorem 3.9 from [14], the approximate solution converges to the solution of problem (1.1), when the time intervals $h = t/n \to 0$.

Let X be a Banach space and we define $\alpha_j : \mathbb{R} \to \mathbb{R}$ a periodic function of period 1 as:

$$\alpha_j(t) = \begin{cases} 3 & \text{, if } k + j/3 \leq t < k + (j+1)/3, \\ 0 & \text{, if } k + (j+1)/3 \leq t < k + j + 1, \end{cases}$$

for $k \in \mathbb{Z}$ and $j = 0, 1, 2$.

Given $h > 0$, we define the function $\alpha_h : \mathbb{R} \to \mathbb{R}$ as $\alpha_{jh}(t) = \alpha_j(t/h)$. Clearly $0 \leq \alpha_{jh} \leq 3$, α_{jh} is h-periodic and its mean value is 1.

We consider $\tau_h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\tau_h(t,t') = \int_{t'}^{t} \alpha_{0h}(t'')dt'',$$

We define $\omega = \{(t,t') \in \mathbb{R}^2 : 0 \leq t' \leq t\}$ and $U_h : \omega \to \mathscr{B}(X)$ given by $U_h(t,t') = U(\tau_h(t,t'))$.

We consider the system,

$$\begin{cases} \partial_t u_h + \alpha_{0h}(t)\sigma(-\partial_{xx})u_h(x,t) = \alpha_{1h}(t)B_1(x,t,u_h(x,t)) + \alpha_{2h}(t)B_2(x,t,u_h(x,t)), \\ u_h(x,0) = u_{h0}(x) \end{cases}$$

where $u(x,t) \in X,\ t > 0,\ \sigma \in \mathbb{C}$ and $B_j : \mathbb{R} \times \mathbb{R} \times X \to X$ is a continuous function with $j = 1, 2$.

Similarly, we define the integral equation:

$$u_h(t) = U_h(t,0)u_{h0} + \int_0^t U_h(t,t')(\alpha_{1h}(t')B_1(x,t,u_h(t')) + \alpha_{2h}(t')B_2(x,t,u_h(t')))dt' \tag{4.1}$$

The following two theorems are developed similar to the results of section 4 of [5], where all results are proved for one nonlinearity. We extend the proofs to consider two nonlinearities. The following Theorem is similar to Propostion 4.3 of [5].

Theorem 4.2. *Let u_h be the solution of* (4.1)*, if $W_{2,k} = u_h(kh+h)$, $V_{0,k} = u_h(kh+h/3)$ and $V_{1,k} = u_h(kh+2h/3)$ then*

$$V_{0,k+1} = U(h)W_{2,k}, \tag{4.2a}$$

$$V_{1,k+1} = N_1(kh+h/3, kh+2h/3, V_{0,k+1}), \tag{4.2b}$$

$$W_{2,k+1} = N_2(kh+2h/3, kh+h, V_{1,k+1}), \tag{4.2c}$$

where $N_j,\ (j = 1,2)$ is the flux associated to:

$$\begin{cases} \partial_t w = \alpha_{jh}(t')B_j(t, w(t)), \\ w(0) = w_0, \end{cases}$$

Proof. For $t_1 \in (0,t)$ it is verified

$$u_h(t) = U_h(t,t_1)u_{h0}(t_1) + \int_{t_1}^t U_h(t,t')(\alpha_{1h}(t')B_1(x,t,u_h(t')) + \alpha_{2h}(t')B_2(x,t,u_h(t')))dt'$$

taking that $t_1 = kh$ and $t = kh + h/3$, we have

$$V_{0,k+1} = U_h(kh+h/3,kh)W_{2,k} + \int_{kh}^{kh+h/3} U_h(t,t')(\alpha_{1h}(t')B_1(x,t,u_h(t')) + \alpha_{2h}(t')B_2(x,t,u_h(t')))dt',$$

since $\alpha_{0h}(t') = 3$ for $t' \in [kh, kh+h/3)$, we have $\tau_h(kh+h/3,kh) = h$ and we get (4.2a). Similarly, $\alpha_{0h}(t') = \alpha_{2h}(t') = 0$ for $t \in [kh+h/3, kh+2h/3)$, then $\tau_h(t, kh+h/3) = 0$ and therefore

$$u_h(t) = V_{0,k+1} + 3\int_{kh+h/3}^{t} B_1(x,t,u_h(t'))dt',$$

evaluating in $t = kh + 2h/3$, we obtain (4.2b). The same process can be done to obtain (4.2c). □

Theorem 4.3. *Let $u \in C([0,T^*),X)$ the be solution of the integral problem*

$$u(t) = U(t)u_0 + \int_0^t U(t-t')B(x,t,u(t'))dt',$$

with $B = B_1 + B_2$ both defined as in (4.1). *Let also $T \in (0,T^*)$ and $\varepsilon > 0$. Then there exists $h^* > 0$ such that if $0 < h < h^*$, then u_h the solution of* (4.1) *with $u_h(x,0) = u_0(x)$, is defined in the interval $[0,T]$ and verifies $\|u(t) - u_h(t)\|_X \leq \varepsilon$ for $t \in [0,T]$.*

Proof. The proof is similar to Theorem 4.4 from [5], considering two distinct nonlinear Lipschitz terms. □

Now, we apply Lemma 2.1 from Section 2 related to linear equation and Lemmas 3.2 and 3.3 from Section 3 related to the control equation. In order to obtain the well-posedness result for the solution $u(t)$ of equation (1.1), we use Theorem 4.3. We denote by $N_1 : \mathbb{R}^+ \times \mathbb{R}^+ \times C_u(\mathbb{R}) \to C_u(\mathbb{R})$ the flow generated by the equation (3.2) as $u(t) = N_1(t,t_0,u_0)$, and similarly $u(t) = N_2(t,t_0,u_0)$ the flow generated by the equation (3.1) defined for $t_0 \leq t < T^*(t_0,u_0)$.

Theorem 4.4. *Let $u_0 \in X^1(\mathbb{R})$, then the solution of satisfies* (1.1) *$u(t) \in C([0,T^*(u_0)),X^1(\mathbb{R}))$ for $t \in (0,T^*(u_0))$.*

Proof. For $t \in [0,T^*(u_0))$, let $n \in \mathbb{N}$, $h = t/n$ and $\{W_{2,k}\}_{0\leq k\leq n}, \{V_{0,k}\}_{1\leq k\leq n}, \{V_{1,k}\}_{1\leq k\leq n}$ be the sequences given by $W_{2,0} = u_0$,

$$\begin{aligned} V_{0,k+1} &= U(h)W_{2,k}, \\ V_{1,k+1} &= N_1(kh+2h/3, kh+h/3, V_{0,k+1}), \\ W_{2,k+1} &= N_2(kh+h, kh+2h/3, V_{1,k+1}), \quad k = 0,\dots,n-1. \end{aligned}$$

We claim that $W_{2,k} \in C([0,T^*(u_0)),X^1(\mathbb{R}))$ for $k = 0,\dots,n$. Clearly, the assertion is true for $k = 0$. If $W_{2,k} \in C([0,T^*(u_0)),X^1(\mathbb{R}))$, from Lemma 2.1, we have $U(h)W_{2,k} \in C([0,T^*(u_0)),X^1(\mathbb{R}))$ and from Lemma 3.3 we can see that

$$V_{1,k+1} = N_1(h, V_{0,k+1}) \in C([0,T^*(u_0)),X^1(\mathbb{R})).$$

Similarly, using Lemma 3.2, we have that

$$W_{2,k+1} = N_2(h, V_{1,k+1}) \in C([0, T^*(u_0)), X^1(\mathbb{R})).$$

By Theorem 4.3 we have that $W_{2,n} \to u(t)$ when $n \to \infty$. As $X^1(\mathbb{R})$ is closed, we obtain the result.

□

Aknowledgments

This work was supported by Universidad Abierta Interamericana (UAI) and CONICET–Argentina.

REFERENCES

[1] I.S. Aranson & L. Kramer. The world of the complex Ginzburg-Landau equation. *Reviews of modern physics*, **74**(1) (2002), 99–143.

[2] D. Battogtokh & A. Mikhailov. Controlling turbulence in the complex Ginzburg-Landau equation. *Physica D: Nonlinear Phenomena*, **90**(1-2) (1996), 84–95.

[3] D. Battogtokh, A. Preusser & A. Mikhailov. Controlling turbulence in the complex Ginzburg-Landau equation II. Two-dimensional systems. *Physica D: Nonlinear Phenomena*, **106**(3-4) (1997), 327–362.

[4] A. Besteiro. A note on dark solitons in nonlinear complex Ginzburg-Landau equations. *Mathematica*, **62-85**(1) (2020), 11–15.

[5] A. Besteiro & D. Rial. Global existence for vector valued fractional reaction-diffusion equations. *Publicacions Matemàtiques*, **96** (2021), 653–680.

[6] J.P. Borgna, M.D. Leo, D. Rial & C.S. de la Vega. General Splitting methods for abstract semilinear evolution equations. *Commun. Math. Sci, Int. Press Boston, Inc.*, **13** (2015), 83–101.

[7] T. Cazenave & A. Haraux. "An Introduction to Semilinear Evolution Equations". Oxford Lecture Ser. Math. Appl., Clarendon Press, Rev ed. (1999).

[8] N. Efremidis, K. Hizanidis, H. Nistazakis, D. Frantzeskakis & B. Malomed. Stabilization of dark solitons in the cubic Ginzburg-Landau equation. *Physical Review E*, **62**(5) (2000), 7410.

[9] K.J. Engel & R. Nagel. "One-parameter semigroups for linear evolution equations", volume 194. Springer Science & Business Media (1999).

[10] C. Gallo *et al.* Schrödinger group on Zhidkov spaces. *Advances in Differential Equations*, **9**(5-6) (2004), 509–538.

[11] J. Ginibre & G. Velo. The Cauchy problem in local spaces for the complex Ginzburg-Landau equation I. Compactness methods. *Physica D: Nonlinear Phenomena*, **95**(3-4) (1996), 191–228.

[12] J. Ginibre & G. Velo. The Cauchy Problem in Local Spaces for the Complex Ginzburg—Landau Equation II. Contraction Methods. *Communications in mathematical physics*, **187**(1) (1997), 45–79.

[13] Y.S. Kivshar & B. Luther-Davies. Dark optical solitons: physics and applications. *Physics reports*, **298**(2-3) (1998), 81–197.

[14] M.D. Leo, D. Rial & C.F.S. de la Vega. High-order time-splitting methods for irreversible equations. *IMA J. Numer. Anal.*, (2015), 1842–1866.

[15] M. Ouzahra. Approximate and exact controllability of a reaction–diffusion equation governed by bilinear control. *European Journal of Control*, **32** (2016), 32–38.

[16] J. Xiao, G. Hu, J. Yang & J. Gao. Controlling turbulence in the complex Ginzburg-Landau equation. *Physical review letters*, **81**(25) (1998), 5552.

[17] P. Zhidkov. The Cauchy problem for the nonlinear Schrödinger equation. Technical report, Joint Inst. for Nuclear Research (1987).