

## Stable Plane-Gauss Maps on Closed Orientable Surfaces

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**ABSTRACT.** The aim of this paper is to study the couple of stable plane Gauss maps  $f = (f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$  from a global point of view, where  $M$  is a smooth closed orientable surface,  $f_2$  is a projection and  $f_3$  is Gauss map. We associate this maps a pair of  $\mathcal{MF}$ -graph. We will study their properties, giving conditions on the graphs that can be realized by pairs of maps with couples from pre-determined singular sets.

**Keywords:** closed surfaces, graphs, stable maps.

### 1 INTRODUCTION

The singular set of stable maps from closed surfaces  $M$  to the plane, according to Whitney theory, consists of a finite number of closed curves in  $M$ , possibly containing isolated cusp points. The apparent contour (i.e. the image of the singular set) consists of a number of immersed curves in  $\mathbb{R}^2$  (possibly with cusps) whose self-intersections are all transverse and disjoint from the cusps. The singular and regular components in the surface  $M$  codify relevant information about the stable map. In [5] graphs with weights on vertices were introduced as global topological invariants for stable maps of type  $f_2 : M \rightarrow \mathbb{R}^2$ . They describe the position of the singular and regular set on  $M$ . In these graphs, edges, vertices and weights corresponds to the singular components, regular components and the genus of the regular components of  $M$ , respectively.

The singularities of a stable Gauss map, in Whitney's sense, being fold curves with isolated cusp points, are called the *parabolic set* of the surface [1, 2, 8]. Each parabolic curve in this set separates a hyperbolic region from an elliptic region of the surface. In order to study the global behaviour of a Gauss map, it is useful to codify the information relative to the complement of the parabolic set on the surface. In [6], the authors introduce the study of graphs with weights

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associated with stable Gauss maps, as global topological invariants, where it is shown that *any weighted bipartite graph can be associated to a stable Gauss map from an appropriate closed orientable surface*.

The purpose of this work is to analyse the topological structure of a closed orientable surface  $M$  immersed in the Euclidean 3-space, from two points of view: the Gaussian curvature of the immersed surface and its projection in the plane. The global study of these two different points of view may present more information than only one of the particular cases and may contribute to classify two-dimensional objects in space.

We denote by  $f_2 : M \rightarrow \mathbb{R}^2$  a map of a surface in the plane and  $f_3 : M \rightarrow \mathbb{S}^2$  a Gauss map of  $M$ . The graphs associated to  $f_2$  and  $f_3$  will be called  $\mathcal{M}$ -graph and  $\mathcal{F}$ -graph, respectively. The pair of graphs associated with stable maps pair  $(f_2, f_3) : M \rightarrow \mathbb{R} \times \mathbb{S}^2$ , it will be called  $\mathcal{MF}$ -graph.

We focus on  $\mathcal{MF}$ -graph when the total weight is equal to zero. We introduce families of graphs that are  $\mathcal{MF}$ -graph and we show that every tree with zero weight can be associated to a stable map  $(f_2, f_3) : \mathbb{S}^2 \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ .

This paper is organized as follows: In Section 2, we will introduce a brief revision of  $\mathcal{M}$ -graph, based on [5] and  $\mathcal{F}$ -graph, defined in [6]. In section 3, we will give a review of codimension one transitions that were introduced in [7] for stable maps between surfaces and [3] for Gauss maps. The main results of the paper will be proved in Section 4 by inductive constructive process starting from simple basic graphs of well-known examples and then apply suitable codimension one transitions. The key point is the existence of pairs of basic graphs with total weight equal to zero and a suitable manipulation of the immersions of a closed orientable surface in Euclidean 3-space, to realize the  $\mathcal{F}$ -graph, so that there is a direction  $v \in \mathbb{R}^3$  in which the projection in the orthogonal plane to  $v$ , of this immersion, realizes the  $\mathcal{M}$ -graph.

## 2 STABLE PLANE-GAUSS MAPS

Let  $M$  and  $N$  be smooth connected closed orientable surfaces and  $f, g : M \rightarrow N$  be smooth maps between them. It is said that  $f$  is  $\mathcal{A}$ -equivalent (or equivalent) to  $g$  if there are orientation-preserving diffeomorphisms,  $k : M \rightarrow M$  and  $l : N \rightarrow N$ , such that  $g \circ k = l \circ f$ . A smooth map  $f : M \rightarrow N$  is said to be *stable* if all maps sufficiently close to  $f$ , in the Whitney  $C^\infty$ -topology (see [4]), are equivalent to  $f$ .

The concept of stability for a Gauss map of a surface immersed in  $\mathbb{R}^3$  is slightly different from the general case of maps between surfaces in the sense that it depends on perturbations of the immersion rather than on those of the map itself.

Let  $j : M \rightarrow \mathbb{R}^3$  be an immersion of  $M$  in  $\mathbb{R}^3$ . Consider the following maps:

1. A stable projection  $p_v : j(M) \rightarrow \mathcal{P} \subset \mathbb{R}^3$ , where  $\mathcal{P}$  is a orthogonal plane to  $v \in \mathbb{R}^3$  and  $p_v$  is the restriction to  $j(M)$ , of the projection of  $\mathbb{R}^3$  in  $\mathcal{P}$ . A diffeomorphism  $g : \mathcal{P} \rightarrow \mathbb{R}^2$  and  $f_2 : M \rightarrow \mathbb{R}^2$  given by  $f_2 = g \circ p_v \circ j$ .

2. Gauss map  $\mathcal{N}_j : j(M) \rightarrow \mathbb{S}^2$  of  $j(M)$  and  $f_3 : M \rightarrow \mathbb{S}^2$  given by  $f_3 = \mathcal{N}_j \circ j$ .

In this case, the map  $f_3$  is called *Gauss map of  $M$* , associated to the immersion  $j$ . The Gauss map  $\mathcal{N}_j$  is said to be *stable* if there exists a neighborhood  $\mathcal{U}_j$  of  $j$  in the space  $\mathcal{I}(M, \mathbb{R}^3)$  of immersions of  $M$  into  $\mathbb{R}^3$  such that for all  $k \in \mathcal{U}_j$ , the Gauss map  $\mathcal{N}_k$  associated to  $k$  is  $\mathcal{A}$ -equivalent to  $\mathcal{N}_j$ .

**Definition 2.1.** We say that the smooth map  $f = (f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$  is a *stable plane Gauss map* if each  $f_i$ ,  $i = 2, 3$ , is a stable map.

Let  $f = (f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$  be a stable plane-Gauss map. A point of the surface  $M$  is a *regular* point of  $f_i$ , ( $i = 2, 3$ ), if the map  $f_i$  is a local diffeomorphism around that point and otherwise *singular*. We denote by  $\Sigma f_i$  the *singular set* of  $f_i$  and its image  $Bf_i = f_i(\Sigma f_i)$  is the *branch set* of  $f_i$ . We observe that regular points of  $f_3$  corresponds, geometrically, to elliptic or hyperbolic points of  $j(M)$ . Singular points of  $f_3$  corresponds to parabolic points of  $j(M)$  (see [6]). By Whitney's Theorem (see [4]),  $\Sigma f_i$  consists of closed curves with fold points, possibly containing isolated cusp points. Then the branch set of  $f_i$ , consists of a collection of closed curves immersed in the target surface with possible isolated cusps and self-intersections (double points). The orientation of the branch set is as follows: transverse a branch curve following the orientation with nearby points on our left that have two more inverse images than those on our right. The non-singular set (which is immersed in the target surface by the map  $f_i$ ) consists of a finite number of regions.

## 2.1 Graphs of stable maps

The singular sets of two equivalent maps are equivalent, in the sense that, there is a diffeomorphism carrying one singular set onto the other and similarly for the branch sets. Thus any invariant diffeomorphism of singular sets or branch sets will automatically be a topological invariant of the map. Both the number of connected components of the singular set and the topological types of the regions are topological invariants. This information may be encoded on a weighted graph  $\mathcal{G}^i(V^i, E^i, W^i)$ , where the pair  $(M, \Sigma f_i)$  may be reconstructed (up to diffeomorphism) (see [5, 6]). On the weighted graph  $\mathcal{G}^i(V^i, E^i, W^i)$  defined by a stable map  $f_i$  each of its  $E^i$  edges corresponds to a path-component of the singular set of  $M$  and each of its  $V^i$  vertices to a different regular region of the surface. An edge is *incident* to a vertex if and only if the corresponding singular curve to the edge lies in the boundary of the regular region corresponding to the vertex (see Figure 1). A *weight* is defined as the genus of the region that represents and it is attached to each vertex. The number  $W^i$  is the sum of weights of all graph vertices.

It is remarkable that  $V^i$  represents the number of connected components of  $M \setminus \Sigma f_i$ ;  $E^i$  the number of connected components of  $\Sigma f_i$  and  $W^i$  the total sum of the genus of the components of  $M \setminus \Sigma f_i$ .

Given orientations of the surfaces  $M$ ,  $\mathbb{R}^2$  and  $\mathbb{S}^2$ , a region of  $M \setminus \Sigma f_2$  is *positive* if the map preserves orientation and *negative* otherwise. A region of  $M \setminus \Sigma f_3$  is *positive* if it has positive Gaussian curvature and *negative* otherwise. A vertex of  $\mathcal{G}^i(V^i, E^i, W^i)$  has a *positive* (or *negative*) label depending on whether the region that it represents is positive (or negative) (see Figure 2).

Since each component of  $\Sigma f_i$  is the boundary of a positive and a negative region, the signs of the vertices are assigned alternatively, hence the graph associated to stable map  $f_i$  is bipartite. We denote by  $V^{i\pm}$  the number of positive (negative) vertices and  $W^{i\pm}$  the total weight associated to the positive (negative) vertices.

The graph  $\mathcal{G}^i(V^i, E^i, W^i)$  is a global invariant and it classifies completely the topology of the regular set of the stable maps  $f_i$ . Moreover, it will be used as a tool for the construction of examples of stable maps between surfaces.

Figure 1 illustrates an embedding of the torus in the 3-space, where  $f_2$  is a map of the torus in the plane and  $f_3$  is a Gauss map. Let us remark that the sets of singular curves of these maps  $f_2$  and  $f_3$  are not equivalent.

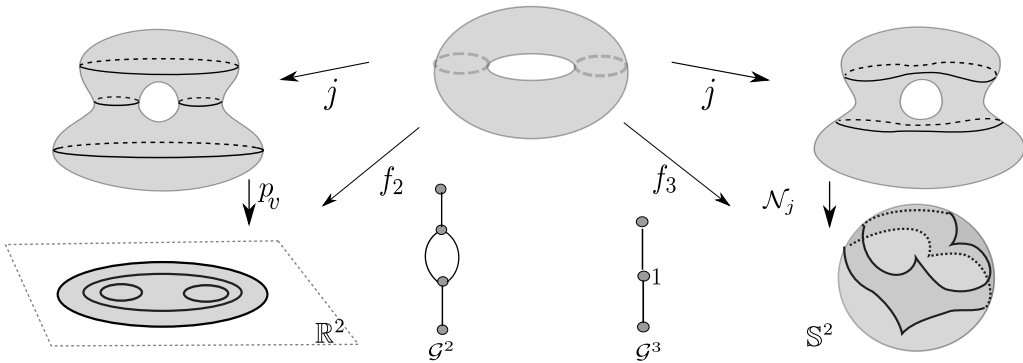


Figure 1: Example of stable plane-Gauss maps.

**Definition 2.2.** Let  $\mathcal{G}$  be a connected graph with non-negative integer weights on its vertices. We say that  $\mathcal{G}$  is a (see Figure 1):

1. *Mendes-graph* or simply  $\mathcal{M}$ -*graph* if exists a smooth connected closed surface  $M$ , a smooth surface  $N$  and a stable map  $f_2 : M \rightarrow N$  such that  $\mathcal{G}$  is the graph of  $f_2$  (in the sense of [5]).
2. *Fuster-graph* or simply  $\mathcal{F}$ -*graph* if exists a smooth connected closed surface  $M$  and an immersion  $j : M \rightarrow \mathbb{R}^3$  whose stable Gauss map  $f_3 : M \rightarrow \mathbb{S}^2$ , associated to  $j$ , has  $\mathcal{G}$  as its associated graph (in the sense of [6]).

In this paper, we consider  $M$  orientable and  $N = \mathbb{R}^2$ . The following result characterizes  $\mathcal{M}$ -graphs and  $\mathcal{F}$ -graphs.

**Theorem 2.1.** [5, 6] *Let  $\mathcal{G} = \mathcal{G}(V, E, W)$  be a weighted graph with  $E > 0$ . The following conditions are equivalent:*

1.  $\mathcal{G}$  is a bipartite graph.

2.  $\mathcal{G}$  is a  $\mathcal{M}$ -graph.
3.  $\mathcal{G}$  is a  $\mathcal{F}$ -graph.

In this case, the genus of  $M$  is given by  $g(M) = 1 - V + E + W$ .

Consequently, any tree with  $W = 0$  is a  $\mathcal{M}$ -graph and a  $\mathcal{F}$ -graph where  $M = \mathbb{S}^2$ .

The Figure 2 illustrates two stable maps  $f_2, g_2 : \mathbb{S}^2 \rightarrow \mathbb{R}^2$ , whose  $\mathcal{M}$ -graphs type  $\mathcal{G}_1^2(6, 5, 0)$  and  $\mathcal{G}_2^2(6, 5, 0)$  are non equivalent, where  $j_i$ 's indicates the respective embedding from  $\mathbb{S}^2$  in  $\mathbb{R}^3$  and  $p_v, v = e_3$ , the canonical projection. This example shows that the  $\mathcal{M}$ -graph is an invariant and apparent contour sets of  $f_2$  and  $g_2$  can not differ.

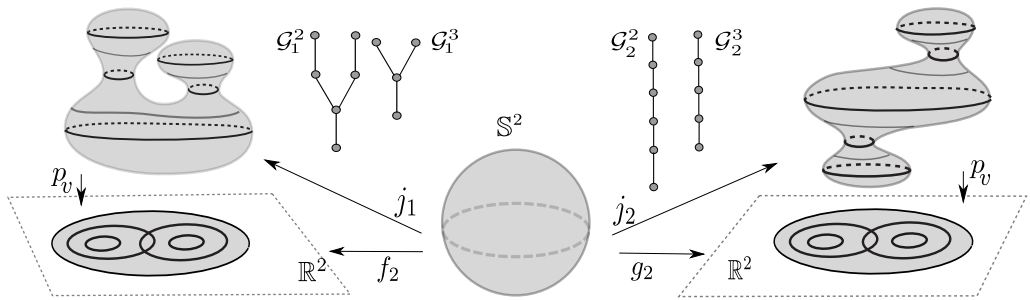


Figure 2: Examples of  $\mathcal{M}$ -graph and  $\mathcal{F}$ -graph type tree.

The Figure 2 illustrates two stable maps  $f_3, g_3 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ , whose  $\mathcal{F}$ -graphs  $\mathcal{G}_1^3(4, 3, 0)$  and  $\mathcal{G}_2^3(5, 4, 0)$  are not equivalent, where  $j_i$  ( $i = 1, 2$ ) indicate the respective embedding from  $\mathbb{S}^2$  in  $\mathbb{R}^3$ .

## 2.2 Graph of stable plane-Gauss maps

**Definition 2.3.** Let  $f = (f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$  be a stable plane-Gauss map. We denote the graph of  $f$  by  $(\mathcal{G}^2, \mathcal{G}^3)$ , where  $\mathcal{G}^2$  is the graph of  $f_2$  and  $\mathcal{G}^3$  the graph of  $f_3$ .

**Definition 2.4.** We say that  $(\mathcal{G}, \mathcal{H})$  is a  $\mathcal{MF}$ -graph if exists a stable plane-Gauss map  $f = (f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$  such that the graph of  $f_2$  is  $\mathcal{G}$  the graph of  $f_3$  is  $\mathcal{H}$ .

Note that a graph  $(\mathcal{G}, \mathcal{H})$  is an  $\mathcal{MF}$ -graph when  $\mathcal{G}$  is a  $\mathcal{M}$ -graph and  $\mathcal{H}$  is a  $\mathcal{F}$ -graph considering the maps  $f_2$  and  $f_3$  are those that realize  $\mathcal{G}$  and  $\mathcal{H}$ , respectively, have the same domain.

The  $\mathcal{M}$ -graph contributes to determining the position of the regular regions and singular curves of  $f_2$  while  $\mathcal{F}$ -graph contributes with the disposition of the elliptic and hyperbolic regions.

Given a closed and oriented surface  $M$ , by Definition 2.3, every stable plane-Gauss map  $(f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$  is associated with an  $\mathcal{MF}$ -graph. Some natural questions can be considered:

- i) Is every graph  $(\mathcal{G}, \mathcal{H})$  an  $\mathcal{MF}$ -graph?
- ii) Otherwise, what conditions should we impose on a graph  $(\mathcal{G}, \mathcal{H})$  in order to be an  $\mathcal{MF}$ -graph?

In Section 4 we will introduce families of  $\mathcal{MF}$ -graphs and we will provide necessary and sufficient conditions for which pairs of graphs with zero weights to be  $\mathcal{MF}$ -graphs. The proofs of the previous results for  $\mathcal{M}$ -graphs and  $\mathcal{F}$ -graphs are based on a convenient manipulation of codimension one transitions (lips and beaks) in the space of stable maps (Section 3) from the examples of maps with a predetermined set.

### 3 LIPS AND BEAKS TRANSITIONS OF PLANE-GAUSS MAPS

We can always obtain a new stable plane-Gauss map, associated with new  $\mathcal{MF}$ -graphs by changing the immersion of  $M$  in  $\mathbb{R}^3$ . These alterations can be made changing the Gauss maps  $f_3 : M \rightarrow \mathbb{S}^2$ , so that they may alter or not the images of the  $f_2 : M \rightarrow \mathbb{R}^2$ , depending on the map in the plane or changing  $f_2$  without varying  $f_3$ .

Here we will give a brief review on the perturbations that modify the graph, called as beaks and lips, introduced in [7], in the case of maps from surface to the plane, and for the case of Gauss maps (see [3] for more details).

Let  $M$  be a smooth connected closed orientable surface and  $N = \mathbb{R}^2, \mathbb{S}^2$  with the usual orientation. Let us denote by  $\mathcal{C}^\infty(M, N)$  and  $\mathcal{E}(M, N) \subset \mathcal{C}^\infty(M, N)$  the spaces of all smooth maps and all smooth stable maps from  $M$  to  $N$ .  $\mathcal{E}(M, N)$  is open and dense in  $\mathcal{C}^\infty(M, N)$ , on Whitney  $C^\infty$  topology. The complement  $\Delta = \mathcal{C}^\infty(M, N) \setminus \mathcal{E}(M, N)$ , is known as *the discriminant set* in  $\mathcal{C}^\infty(M, N)$ .

A *codimension one transition* corresponds to the intersection of  $\Delta$  with generic isotopy from a given stable map  $f_i$  to another stable map  $g_i$  ( $i = 2, 3$ ) that are in different path-components of  $\mathcal{E}(M, N)$ . In other words, this means that codimension one transition is the point at which a path between  $f_i$  and  $g_i$  transversely intersects a strata of  $\Delta$ . The types of transitions are described in [7], in the case of surfaces in the plane and the case of Gauss maps [3, 6].

In this paper, we are interested in those codimension one transitions affecting graphs of the stable plane-Gauss maps (in addition to change the topology of singular and regular sets), namely, the beaks and the lips transitions (see [5, 6]). These transitions always change the number of cusps.

For lips and beaks transitions that changes the number of cusps of each  $f_i$ , the edge number of the graph, the number of vertices or the weight of the graphs, fulfills  $g(M) = 1 - V^i + E^i + W^i$  ( $i = 2, 3$ ) is constant. For more details, see [5, 6].

**Definition 3.5.** A cusp point  $x \in \Sigma f_i$  ( $i = 2, 3$ ) is called *positive* (resp. *negative*) if its local mapping degree, in a neighborhood  $U_x$  of  $x$  is  $+1$  (resp.  $-1$ ) with respect to the given orientations.

- i) **Lips transition  $L_i^\pm$  of map  $f_i$  ( $i = 2, 3$ ):** increases the number of cusps and increases by 1 the number of singular curves of  $f_i$ :  $L_i^+$  increases by 1 the number of regions negative and  $L_i^-$  increases by 1 the number of regions positive. Consequently,  $L_i^\pm$  increases one vertex and one edge on the graph  $\mathcal{G}^i$ .
- ii) **Beaks transition  $B_i$  of map  $f_i$  ( $i = 2, 3$ ):** increases the number of cusps. Beaks transitions of  $f_i$  can be classified in eight different transitions that change the graph  $\mathcal{G}^i$ , are illustrated (locally) in Figure 3, where in the picture  $X, X_1, Y, Z, Z_1$  and  $Z_2$  denote (locally) the regions that hold the transitions and the numbers 1 and 2 represents the number of singular curves:
  - $B_{vi}^{+,\pm}$ : increases by 1 the number of vertices in  $V^{i\pm}$  and the number of edges  $E^i$ .
  - $B_{vi}^{-,\pm}$ : decreases by 1 the number of vertices in  $V^{i\pm}$  and the number of edges  $E^i$ .
  - $B_{wi}^{+,\pm}$ : increases by 1 the weight in  $W^{i\pm}$  and decreases by 1 the number  $E^i$ .
  - $B_{wi}^{-,\pm}$ : decreases by 1 the weight in  $W^{i\pm}$  and increases by 1 the number  $E^i$ .

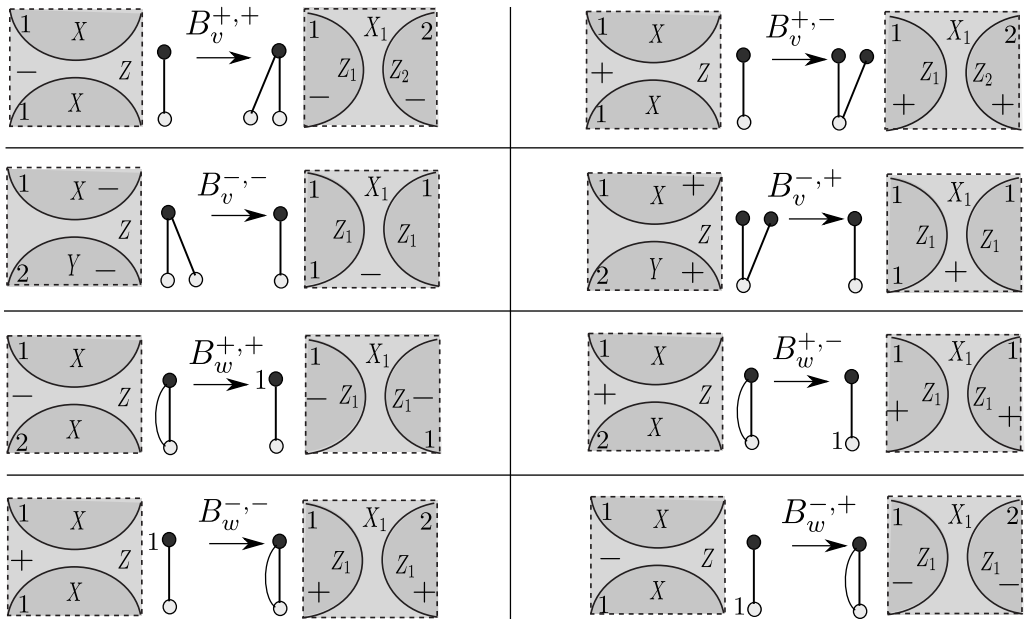


Figure 3: Decomposition of beaks transition. The index  $i$  ( $i = 2, 3$ ) was omitted.

A transition  $\delta \in \{L_i^\pm, B_{vi}^{\pm,\pm}, B_{wi}^{\pm,\pm}\}$  that decreases the number of cusps, will be denoted by  $-\delta$ .

Figure 4 illustrates a sequence of transitions on the sphere that simultaneously change the  $\mathcal{M}$ -graph and the  $\mathcal{F}$ -graph.

The beaks and lips transitions for maps in the plane, can change or not the  $\mathcal{F}$ -graph (see Figure 4 and Figure 6). Similarly, beaks and lips for Gauss maps can change or not the  $\mathcal{M}$ -graph, as illustrates the Figure 5.

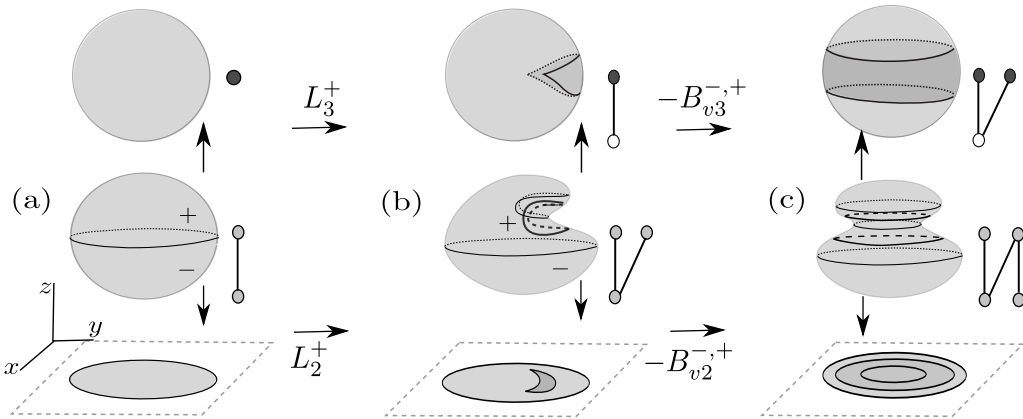


Figure 4: Transitions that change the  $\mathcal{M}$ -graph and  $\mathcal{F}$ -graph.

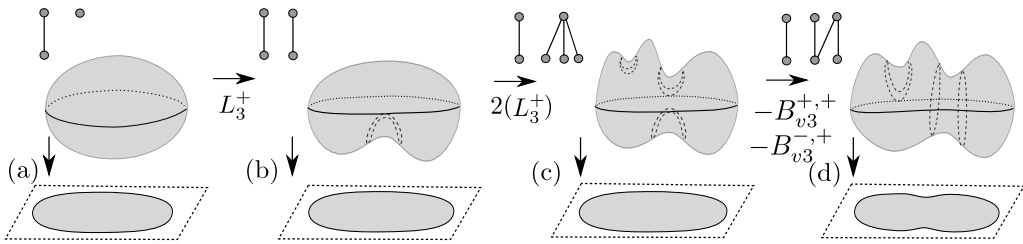


Figure 5: Transitions that only change the  $\mathcal{F}$ -graph.

Note that the sequence of transitions, in Figure 5, does not change the  $\mathcal{M}$ -graph it also does not change the apparent contour. This shows that the graph of the Gaussian map is an invariant that helps to refine the classification of maps in the plane.

#### 4 REALIZATION OF SOME GRAPHS WITH TOTAL WEIGHT ZERO

In this section we present some families of graphs  $(\mathcal{G}^2, \mathcal{G}^3)$  that are  $\mathcal{M}\mathcal{F}$ -graphs, that is, graphs that can be realized by stable plane-Gauss maps in the sense that there is a smooth connected closed orientable surface  $M$  and a stable plane-Gauss map  $f = (f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$  such that the graph of  $f$  is  $(\mathcal{G}^2, \mathcal{G}^3)$ . This will be done through beaks transitions and lips on smaller maps already known. The results here will only be valid for zero weight graphs. For weight greater than zero will be treated in later works.

##### 4.1 Examples

Before introducing the general result tree-type graphs (which satisfies  $V = E + 1$ ) with zero weight and some graphs with cycles in general, we will see some examples of realization.



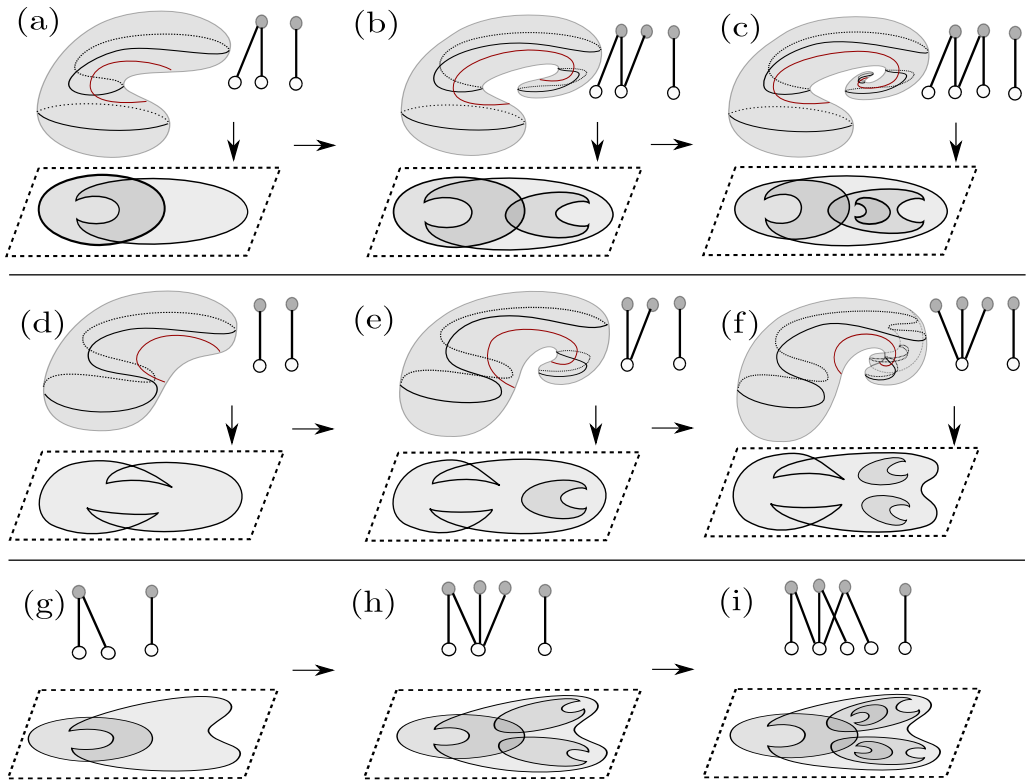


Figure 6: Realization of some  $\mathcal{MF}$ -graphs type tree.

$(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(1, 0, 0))$  can be realized (see Figure 5-(a)) by a  $S^2$  embedding in the 3-space with all elliptic points, which projected in a plane has a unique singular curve, which separates two regular regions.

$(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(2, 1, 0))$  can be realized as Figure 5-(b), by lip transition  $L_3^+$  on (a) on one of the regular regions  $f_2$ , without changing  $\Sigma f_2$ .

**Definition 4.6.** The *degree* of a vertex  $v$  in a graph is the number of edges incident to it. A tree it is called *star positive (negative)* if it has  $V - 1$  negative vertices (positive) with degree 1.

Figure 6 illustrates a sequence of lips transitions  $L_2$  that changes the graph  $\mathcal{G}^2$  without changing the graph  $\mathcal{G}^3$ .

In (a) we see an immersion with a parabolic curve (two cusp points) allows a projection on the plane with two singular curves (two cusp points) two negative and positive regions. From (a) to (b) the transition  $L_2^+$  adds a singular curve in the projection on the plane (creating two negative cusps) and stretching the parabolic curve and the hyperbolic and elliptical regions. (c) is analogous to (b). Following this transition line, it can be obtained a map with graph  $(\mathcal{G}^2(V, V - 1, 0), \mathcal{G}^3(2, 1, 0))$ , where  $\mathcal{G}^2$  has  $V - 2$  vertices with degree 2.

Figure 6-(f) illustrates realization of star graph  $(\mathcal{G}^2(4, 3, 0), \mathcal{G}^3(2, 1, 0))$ , as follows: In (d) we see an immersion with a parabolic curve (two points of cusps) and a projection in the plane with the only singular curve with four cusps, it can be obtained from (a) by a transition beaks  $B_{v_2}^{-,-}$ . From (d) to (e) the transition  $L_2^+$  adds a singular curve in the projection on the plane and stretching the parabolic curve and the hyperbolic and elliptical regions. From (e) to (f) the transition  $L_2^+$  (in the same region) adds another singular curve to the projection in the plane. Following this transition line, it can be obtained maps with graph  $(\mathcal{G}^2(V, V - 1, 0), \mathcal{G}^3(2, 1, 0))$ , where  $\mathcal{G}^2$  is a negative star.

To realize a graph  $\mathcal{G}^2$  with a vertex  $v$  of degree  $m > 2$ , we can apply  $m$  lips transtions  $L_2^\pm$  in the same region that corresponds to  $v$  (see sequence (g) to (i) Figure 6) stretching spiral type in parallel, without changing the topology of the parabolic set and the graph  $\mathcal{G}^3$ .

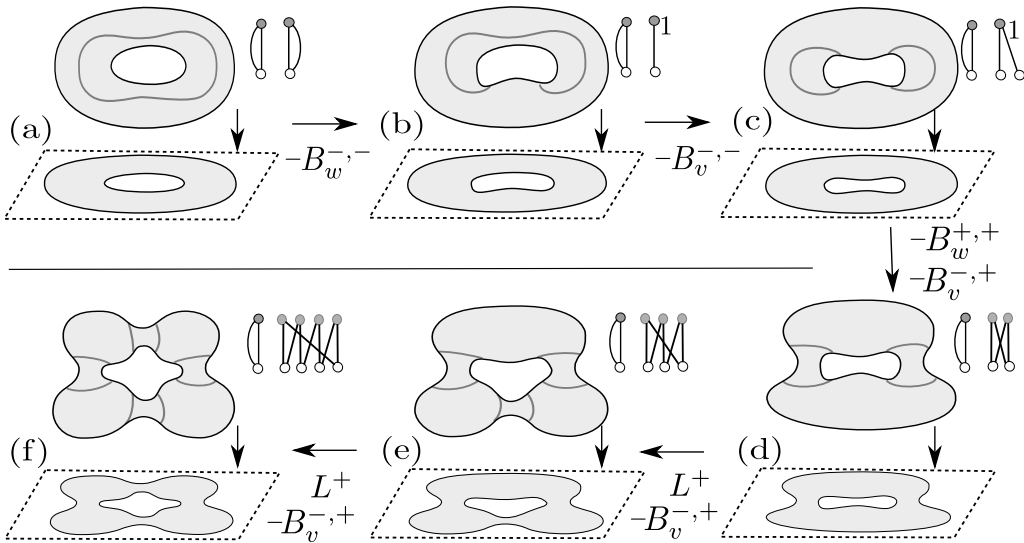


Figure 7: Example of  $\mathcal{MF}$ -graphs with cycles.

Figure 7 shows a sequence of stable maps of the torus with their graphs with transitions that changes the topology of parabolic curves keeping invariant the topology of map curves in the plane. In: (a)  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(2, 2, 0))$ , each parabolic curve of  $f_3$  has 4 cusp points: two positive and two negative (see [6]); (b) is obtained from (a) by transitions  $-B_{w_3}^{-,-}$ ; it removes the pairs of positive cusps of  $f_3$ , leaving the new map with a elliptic region of genus one and one hyperbolic regions simply connected; (c) is obtained from (b) by transitions  $-B_{v_3}^{-,-}$ ; dividing the parabolic curve and the hyperbolic region into two components; (d) is obtained from (c) by transitions  $-B_{w_3}^{+,+}$ , which decomposes a parabolic curve and removes the genus from the elliptic region and is followed by  $-B_{v_3}^{-,+}$  which decomposes the following parabolic curve, obtaining a map with 4 parabolic curves that separates two elliptic regions from two hyperbolic regions, all homeomorphic to the cylinder and with zero weight graph  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(4, 4, 0))$ ; (e) is obtained from (d) by transitions  $L_3^+$  followed by  $-B_{v_3}^{-,+}$ , creating a new elliptic and hyperbolic region with the

graph  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(6, 6, 0))$ ; the same happens with the transitions from (e) to (f) that realizes the graph  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(8, 8, 0))$ .

We can obtain a stable map that realizes the graph  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(V, V, 0))$ , without changing the  $\mathcal{M}$ -graph in the following way (see Figure 7-(e) and (f)).

Figure 8 shows the construction of an immersion of the torus and 4-torus that admits a projection on the plane, which realizes the respective graphs:

$$(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(2, 2, 0)), (\mathcal{G}^2(2, 3, 0), \mathcal{G}^3(2, 3, 0)) \text{ and } (\mathcal{G}^2(2, 5, 0), \mathcal{G}^3(2, 5, 0)).$$

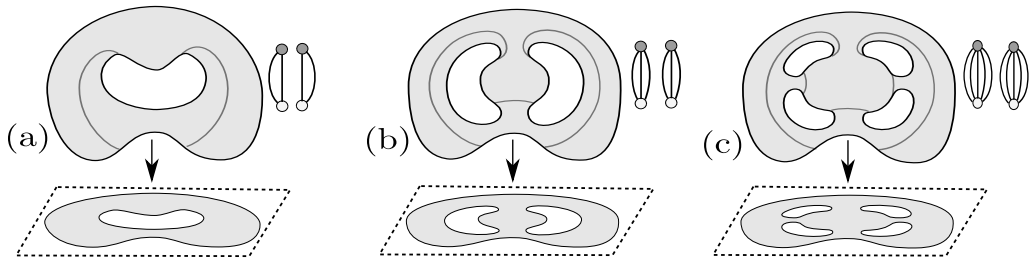


Figure 8: Example of the realization of  $(\mathcal{G}^2(2, K, 0), \mathcal{G}^3(2, K, 0))(k = 2, 3, 5)$ .

### 4.2 Tree realization

**Theorem 4.2.**  $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(V, V - 1, 0))$  is an  $\mathcal{MF}$ -graph, for all  $V > 0$ .

**Proof.** We have shown that  $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(1, 0, 0))$  is an  $\mathcal{MF}$ -graph (see Figure 5). To realize the couple  $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(V, V - 1, 0))$  remove a negative (or positive) vertex, of degree 1, from  $\mathcal{G}^3(V, V - 1, 0)$  and suppose the pair  $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(V - 1, V - 2, 0))$  realizes by a map  $(g_2, g_3) : \mathbb{S}^2 \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ . Through the lips transition  $L_3^+$  (or  $L_3^-$ ) on  $(g_2, g_3)$ , without changing the  $\Sigma f_2$ , we can get a map  $(f_2, f_3)$  which realizes the pair with  $\mathcal{G}^3(V, V - 1, 0)$ .  $\square$

**Corollary 4.2.**  $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(V, V - 1, 0))$ , where  $\mathcal{G}^3$  is a star with  $V > 0$ , is an  $\mathcal{MF}$ -graph.

**Theorem 4.3.** Every graph  $(\mathcal{G}^2(V, V - 1, 0), \mathcal{G}^3(2, 1, 0))$ , with  $V > 1$ , is an  $\mathcal{MF}$ -graph.

**Proof.** We have shown that  $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(2, 1, 0))$  is an  $\mathcal{MF}$ -graph (see Figure 5). To realize the couple  $(\mathcal{G}^2(V, V - 1, 0), \mathcal{G}^3(2, 1, 0))$  remove a negative (or positive) vertex, of degree 1, from  $\mathcal{G}^2(V, V - 1, 0)$  and suppose the pair  $(\mathcal{G}^2(V - 1, V - 2, 0), \mathcal{G}^3(2, 1, 0))$  realizes by a map  $(g_2, g_3) : \mathbb{S}^2 \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ . Through the lips transition  $L_2^+$  (or  $L_2^-$ ) on  $(g_2, g_3)$ , without changing the  $\Sigma f_3$  (as Figure 6), we can get a map  $(f_2, f_3)$  which realizes the pair with  $\mathcal{G}^2(V, V - 1, 0)$ .  $\square$

**Corollary 4.3.**  $(\mathcal{G}^2(V, V - 1, 0), \mathcal{G}^3(2, 1, 0))$ , where  $\mathcal{G}^2$  is a star with  $V > 1$ , is an  $\mathcal{MF}$ -graph.

Follows from Theorems 4.2 and 4.3 the next result:

**Theorem 4.4.** All graph  $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$ , with  $V^2, V^3 > 1$ , is an  $\mathcal{MF}$ -graph.

**Proof.**  $(\mathcal{G}^2(2, 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$  can be realized by a map  $(f_2^0, f_3)$  as Theorem 4.2. Then, chooses the hyperbolic and elliptical regions corresponding to vertices the  $u_i$  and  $v_i$  in the neighborhood of the parabolic curve associated with edges  $u_i v_i$ , of  $\mathcal{G}^3(V^3, V^3 - 1, 0)$ . Lips transitions can be applied to one of the regular regions, as in Theorem 4.3, conveniently, to obtain a map  $(f_2, f_3)$  that realizes the graph  $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$ , without changing the topology of the singular set of  $f_3$ . □

The graph  $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(1, 0, 0))$  is realizable only with  $V^2 = 2$ , that is, to realize the graph  $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$ , with  $V^2 > 2$  we must have  $V^3 > 1$ .

### 4.3 Graphs with cycles

We present some families of graphs with cycles that can be obtained by beaks and lips transitions from already known maps.

**Lemma 4.1.** All bipartite graph  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(V, V, 0))$  is an  $\mathcal{MF}$ -graph.

**Proof.** Let  $\mathcal{G}^3(V', V', 0)$  be a subgraph of  $\mathcal{G}^3(V, V, 0)$ , where all vertices belong to the graph cycle.  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(V', V', 0))$  can be realized as Figure 7.

The vertices and edges outside of  $\mathcal{G}^3(V', V', 0)$  can be realized by transitions  $L_3^+$  followed by  $-B_{w_3}^{-,+}$ , if it is required, new elliptic and hyperbolic regions can be created as in Theorem 4.2, without changing the topology of the singular set of  $f_2$ . □

**Proposition 4.1.** All bipartite graph  $(\mathcal{G}^2(V^2, V^2, 0), \mathcal{G}^3(V^3, V^3, 0))$ , where  $\mathcal{G}^2$  has the cycle with two vertices, is an  $\mathcal{MF}$ -graph.

**Proof.** By Lemma 4.1, the graph  $(\mathcal{G}^2(2, 2, 0), \mathcal{G}^3(V^3, V^3, 0))$  is realized by a torus map. The graph  $(\mathcal{G}^2(V^2, V^2, 0), \mathcal{G}^3(V^3, V^3, 0))$  can be realized, as Theorem 4.4, by transitions on  $f_2$  that does not alter the topology of the singular set of  $f_3$  (see Figure 6). □

**Lemma 4.2.**  $(\mathcal{G}^2(2, K, 0), \mathcal{G}^3(2, K, 0))$  is an  $\mathcal{MF}$ -graph.

**Proof.** To realize this graph, just immerse the  $K - 1$ -torus as Figure 8, with  $K$  parabolic curves that separate an elliptical and a hyperbolic region, both homeomorphic to discs with  $K - 1$  holes, has projection with  $K$  singular curves that separate two regular regions also homeomorphic discs with  $K - 1$  holes. □

**Proposition 4.2.**  $(\mathcal{G}^2(2 + m, K + m, 0), \mathcal{G}^3(2 + n, K + n, 0))$ , where all cycles of  $\mathcal{G}^2$  has length 2, is an  $\mathcal{MF}$ -graph.

**Proof.** For  $m = n = 0$ , it can be realized by Lemma 4.2 by some  $(f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ , with  $g(M) = K - 1$ . By beaks transitions that do not change the topology of the singular set of  $f_2$ , as

Lemma 4.1, can realize  $(\mathcal{G}^2(2, K, 0), \mathcal{G}^3(2 + n, K + n, 0))$  by  $(f_2^1, f_3^1)$ . And applying beaks and lips transitions, as in the proof of Theorem 4.4, can realize  $(\mathcal{G}^2(2 + m, K + m, 0), \mathcal{G}^3(2 + n, K + n, 0))$  from  $(f_2^1, f_3^1)$ .  $\square$

**Lemma 4.3.** *All bipartite graph type  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K, 2K, 0))$ , where  $\mathcal{G}^3$  has at least  $K$  vertices with degree 2 (type daisy), is an  $\mathcal{MF}$ -graph.*

**Proof.** The graph  $\mathcal{G}^3(1 + K, 2K, 0)$  can be realized by a stable Gauss map associated to an immersion of  $K$ -torus with  $2K$  parabolic curves, that separates  $K$  hyperbolic (cylindrical) regions of an elliptical region isomorphic to the disc with  $2K$  holes) (see [6] and Figure 7-(a)). This map allows a projection on the plane with  $K + 1$  singular curves that separates two regular regions, associated with the graph  $\mathcal{G}^2(2, 1 + K, 0)$ . Therefore, we obtain a map  $(f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ , that realizes  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K, 2K, 0))$ .  $\square$

**Proposition 4.3.** *All bipartite graph  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K + m, 2K + m, 0))$ , where all cycle of  $\mathcal{G}^3$  has at most one vertex with a degree greater than 2 is an  $\mathcal{MF}$ -graph.*

**Proof.** First remove all vertices from  $\mathcal{G}^3$  that does not belong to any cycle, obtaining the subgraph  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K + s, 2K + s, 0))$  ( $s \leq m$ ). Then take a map  $(f_2^0, f_3^0)$  that realizes the daisy graph  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K, 2K, 0))$  as Lemma 4.3. Analog to Lemma 4.2, by beaks and lips transitions on  $(f_2^0, f_3^0) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ , where  $M$  has genus  $K$ , that changes the length of the  $\mathcal{G}^3$  without changing the topology of the singular set of  $\mathcal{G}^2$  (see Figure 7), can obtain a map  $(f_2^1, f_3^1)$  on  $M$ , that realizes the graph  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K + s, 2K + s, 0))$ .

Finally, the graph  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K + m, 2K + m, 0))$  can be realized for some stable map  $(f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ , applying  $m - s$  transitions,  $L_3^+$  (and beaks if it is necessary) on  $f_3$ , without changing the  $\mathcal{M}$ -graph, as Lemma 4.2.  $\square$

**Proposition 4.4.** *All bipartite graph type  $(\mathcal{G}^2(2 + m, 1 + K + m, 0), \mathcal{G}^3(1 + K, 2K, 0))$ , where all cycles of  $\mathcal{G}^2$  has length 2, is a  $\mathcal{MF}$ -graph.*

**Proof.**  $(\mathcal{G}^2(2, 1 + K, 0), \mathcal{G}^3(1 + K, 2K, 0))$  can be realized by Lemma 4.3, for some stable map  $(f_2, f_3) : M \rightarrow \mathbb{R}^2 \times \mathbb{S}^2$ , where  $M$  has genus  $K$ . Applying  $m$  transitions,  $L_2^+$  on  $f_2$ , as Theorem 4.4 we can realize the graph  $\mathcal{G}^2(1 + K + m, 1 + K + m, 0)$  without changing the  $\mathcal{F}$ -graph.  $\square$

**Theorem 4.5.** *All graph that can be obtained by extending the cycles (increasing the number of vertices and edges in the cycles) of  $\mathcal{G}^3$  of an  $\mathcal{MF}$ -graph it is also an  $\mathcal{MF}$ -graph.*

**Proof.** Analogous to example in Figure 7.  $\square$

**Theorem 4.6.** *All the graphs that can be obtained by extending vertices and edges outside the graph cycles  $\mathcal{G}^2$  and  $\mathcal{G}^3$  of a  $\mathcal{MF}$ -graph are also an  $\mathcal{MF}$ -graph.*

**Proof.** Analogous to the proof of Theorem 4.4.  $\square$

## 5 CONCLUSIONS

In this work, we first show that all pair of trees  $(\mathcal{G}^2(V^2, V^2 - 1, 0), \mathcal{G}^3(V^3, V^3 - 1, 0))$ , with  $V^2, V^3 > 1$ , can be associated to a stable plane-Gauss map  $f = (f_2, f_3) : \mathbb{S}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{S}^2$ .

Applying codimension one transitions over maps from  $n$ -torus, we show that some graphs  $(\mathcal{G}, \mathcal{H})$ , with  $n$  cycles and total weight equal to zero, are  $\mathcal{MF}$ -graphs.

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