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On the Wildfire Propagation in Flat Lands

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ABSTRACT. Due to global warming, wildfires became more severe and frequent than in the past, and more complex mathematics is needed to provide the model of propagation and predict the behavior of the fire. In this work, for a wildfire spreading in some flat land of zero slope, by using Randers metric the equations of wavefronts and wave rays are provided which lead to presenting a more precise and reliable model of propagation. By considering some hypothetical fire in Boa Nova National Park as an example, a simulation is implemented.

Keywords: Randers metric, Huygens' principle, wavefront, wave rays, wildfire propagation.

1 INTRODUCTION

Providing a more reliable and precise model of wildfire propagation leads to reducing the financial losses to the human properties and damages to the wildlife. Randers geometry is a strong tool to study and model real world problems [9]. In fact, by using this geometry, one is able to provide the equations of paths of fire particles, which are called the *wave rays* in this work, and also the equations of the wavefronts. The Randers geometry is a recently applied method in the process of predicting the spread of wildfire and, generally, the propagation of waves [1, 8, 11]. Very recently, some methods for the propagation of fire waves [5, 6] and water waves [7] are presented.

Another method to provide the mode of propagation is using the simulators such as FARSITE [12]. The methodology in FARSITE is locating some ellipse, as the spherical wavefront, on the perimeter of a given wavefront and then applying the Huygens' principle [2] to find the next wavefront. However, in this methodology the ellipses do not coincide with the spherical wavefronts in reality and, therefore, the wavefront provided by the simulator has some deviation from the waterfront in reality [11]. Consequently, in several cases, the provided model of propagation is not precise and reliable enough. Specially, when the space has some curvature, the real spherical wavefronts are some closed curves far from being ellipses. The curvature of the space

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corresponds to the cases that the fuel, temperature, humidity, or etc., are not constant across the space.

Throughout this work, we assume that some wildfire is spreading across some flat land M which is of zero slope. It is assumed that M is some smooth manifold of dimension 2, for instance M is some open subset of $\mathbb{R}^2$. Some wildfire is spreading across M and the fuel, temperature, humidity, etc., are distributed smoothly throughout M. A constant wind W is spreading across the space such that $h(W,W) < 1$, where h is the Riemannian metric associated with the Randers metric on the space. We suppose that the fire does not create singularities or cut loci, that is no two particles of fire meet. We present the model of propagation by providing the equations of wavefronts and wave rays. For this, we start with some rotated ellipse whose diagonals a and b and angle of rotation α are determined from the experimental data and laboratory. From this ellipse, we find the Riemannian metric h and next, by Theorem 3.4, we find the equations of wavefront and wave rays.

The remainder of this paper is organized as follows. Some preliminaries are provided in Section 2 making the work self-contained. In Section 3, we give the main results of this work. In Section 4, an example in which we simulate a hypothetical wildfire in *Boa Nova National Park* is presented.

2 PRELIMINARIES

Assume that M is a 2-dimensional smooth manifold, for instance it is an open set of $\mathbb{R}^2$, $p = (x_1, x_2)$ a point of it and T_pM the space tangent at point p. Assume that $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ is the canonical basis of T_pM and $V = (u_1, u_2) \in T_pM$ presentation of a vector V according to this basis. A *Riemannian metric* on M is a smooth function h such that it assigns to each point $p \in M$ a positive-definite inner product $h_p : T_pM \times T_pM \to \mathbb{R}$. Given M and a smooth vector field W on it such that $h(W,W) < 1$, the function $F : T_pM \to \mathbb{R}$

$$F(V) = \frac{\sqrt{h^2(W,V) + \lambda h(V,V)}}{\lambda} - \frac{h(W,V)}{\lambda}, \tag{2.1}$$

where $\lambda = 1 - h(W,W)$, is called the *Randers metric*. We should note that some authors use the notation (h,W), instead of F, to denote a Randers metric, see for instance Section 3 of [9]. One shows that each Randers metric can be written as $F = \alpha + \beta$, where α is a Riemannain metric and β is a 1-form on M [4].

Given any Randers space (M,F) and some piece-wise smooth curve $\gamma : [a,b] \longrightarrow M$, the length of γ is $L[\gamma] := \int_a^b F(\gamma'(t))dt$. Given any two points $p, q \in M$, the distance from p to q is defined as

$$d(p,q) := \inf_{\gamma} \int_a^b F(\gamma'(t))dt, \tag{2.2}$$

where the infimum is taken over all piece-wise smooth curves $\gamma : [a,b] \longrightarrow M$ joining p to q. A smooth curve is called a *Randers geodesic* (F-geodesic) if it is locally the shortest time path connecting any two nearby points on this curve. One defines the *Riemannian geodesic* (h-geodesic) in the same way.

For a local coordinate system $(x^i, \frac{\partial}{\partial x^i})$, the Riemannian geodesics are solutions of the following system of equations

$$\frac{d^2x^i}{dt^2}+\frac{1}{2}\sum_{l,k,j=1}^{2} h^{il}\left(\frac{\partial h_{lk}}{\partial x^j}+\frac{\partial h_{lj}}{\partial x^k}-\frac{\partial h_{kj}}{\partial x^l}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}, \qquad i=1,2, \tag{2.3}$$

where $[h_{ij}]$ is the matrix of the Riemannian metric h and $[h^{ij}]=[h_{ij}]^{-1}$ [10].

Given some smooth vector field W on M, the flow of W is the smooth map $\varphi:(-\varepsilon,\varepsilon)\times M\longrightarrow M$ such that for all $p\in M$, $\varphi^p(t):=\varphi(t,p)$ is an integral curve of W; that is [10]

$$\frac{d\varphi^p}{dt}(t)|_{t=0}=W(p). \tag{2.4}$$

A vector field W in a Riemannian space is called *Killing* if and only if, in the local coordinate system $(x^i, \frac{\partial}{\partial x^i})$, [10]

$$\sum_{i,j,k=1}^{2}\left(W_k\frac{\partial h_{ij}}{\partial x^k}+h_{kj}\frac{\partial W_k}{\partial x^i}+h_{ik}\frac{\partial W_k}{\partial x^j}\right)=0, \tag{2.5}$$

where $W=(W_1,W_2)$. Equivalently, W is *Killing* if $\varphi_t: M\longrightarrow M$ is an isometry for all t. That is $d(\varphi_t)_p: T_pM\longrightarrow T_pM$ preserves the metric.

Let S be some source that emits waves. Given any time t, the set of all points of space to which the wave reaches at time t is called the *wavefront* at time t [3]. The source S might be some point, plane, curve, etc.. If S is a single point, the wavefront at time t is called the *spherical wavefront* of radius t and a wavefront at time 1 is called simply a spherical wavefront. Given any wavefront, each point on this wavefront can be seen as a new source which emits spherical wavefront. For this wavefront, the curve tangent to each of the spherical wavefronts is called *envelope* of the wavefront. By *wave rays* of some propagation we mean the paths of particles of fire. The *Huygens' principle* is some important result that states the relation between envelope of a wavefront and the next wavefront as follows.

Theorem 2.1 (Huygens principle). *[3] Let $\phi_p(t)$ be the wavefront of the point p after time t. For every point q of this wavefront, consider the wavefront after time s, i.e. $\phi_q(s)$. Then, the wavefront of point p after time $s+t$, $\phi_p(s+t)$, will be the envelope of wavefronts $\phi_q(s)$, for $q\in\phi_p(t)$.*

In the next result, the equation of the spherical wavefront, for the case that a wildfire is spreading in some flat land with the uniform distribution of the fuel, temperature, humidity, and etc., is provided.

Theorem 2.2. *[6] Assume that a wildfire is spreading in some space M while the fuel has been distributed uniformly, temperature, and humidity are constant across M, the constant wind $W=(0,W_2,W_3)$ is blowing across M, and A is the wavefront at time 0. Then, given any point p in M, the spherical wavefront at p is*

$$Q(u,v,w)+W+p,$$

where

$$Q(u,v,w) = (\frac{u}{a})^2 + (\frac{v\cos\alpha - w\sin\alpha}{b})^2 + (\frac{v\sin\alpha + w\cos\alpha}{c})^2 = 1,$$

were a, b, c, and α are constant real numbers and will be determined from the experimental data. One writes the 2-dimensional version of Theorem 2.2, that is for a propagation on a flat land in dimension 2 with the same hypotheses as those of Theorem 2.2, as below.

Lemma 2.1. *Assume that a wildfire is spreading in some space M while the fuel has been distributed uniformly, temperature, and humidity are constant, the wind $W = (W_1, W_2)$ is blowing across M and A is the wavefront at time 0. Then, given any point p in M, the spherical wavefront and center p is*

$$Q(u,v) + W + p, \tag{2.6}$$

where $Q(u,v)$ is given by the Eq. (2.7).

$$Q(u,v) = (\frac{u\cos\alpha - v\sin\alpha}{a})^2 + (\frac{u\sin\alpha + v\cos\alpha}{b})^2 = 1, \tag{2.7}$$

where a, b, and α are constant real numbers and will be determined from the experimental data. Here α is the angle of rotation about the origin.

In the next theorem, the equations of wave rays and wavefronts are provided for the general case of the wildfire propagating in some space under the influence of some wind that is also *Killing*.

Theorem 2.3. *[6] Assume that a wildfire is spreading in the space M, the wind W, which is a Killing vector filed, is blowing across M, the fuel has been distributed smoothly, and A is the wavefront at time 0. Then:*

(i) Given any point p in A, the wave rays emanating from p are $\gamma_F(t) := \varphi(t, \gamma_h(t))$, where φ is the flow of W and γ_h is the h-geodesic such that $\gamma_h(0) = p$, $|\gamma_h'(t)|_h = 1$, and $d\varphi_p \gamma_h'(0)$ if orthogonal to A.

(ii) The spherical wavefront at time τ and center of some point $p \in M$ is the set
$\{\varphi(\tau, \gamma_h(\tau)) : \gamma_h$ is the unit speed h-geodesic that $\gamma_h(0) = p\}$.

3 FINDING THE EQUATIONS OF WAVEFRONTS AND WAVE RAYS

In this section, we provide the equations of wavefronts and wave rays of propagation. It is assumed that a wildfire is spreading across some flat land M which is of zero slope, the fuel has been distributed smoothly across it and other conditions such as the temperature, humidity, and etc. are changing smoothly across M. The fire starts from some point which is considered as the origin of the coordinate system. Here each point p of M is shown with $p = (x,y) = (x_1,x_2)$ and each point of T_pM is shown with $(u,v) = (u_1,u_2)$.

Theorem 3.4. *Assume that a wildfire is spreading in some flat land of zero slope M while the constant wind $W = (W_1, W_2)$ is blowing across M. If*

$$\sum_{i,j,k=1}^{2} W_k \frac{\partial h_{ij}}{\partial x_k} = 0, \tag{3.1}$$

where $[h_{ij}] = \frac{1}{2}[\frac{\partial^2 Q}{\partial u_i \partial u_j}]$ and

$$Q = (\frac{u\cos\alpha - v\sin\alpha}{a})^2 + (\frac{u\sin\alpha + v\cos\alpha}{b})^2 = 1, \tag{3.2}$$

in which a, b, and α are smooth functions on M and determined from experimental data, then the equation of the wave ray igniting from any point $p \in M$ is given by $\gamma_F(t) = Wt + \gamma_h(t)$, in which $\gamma_h(t)$ is the solution of system of Eqs. (2.3) with the initial conditions $\gamma_h(0) = p$ and $h(\gamma_h{}'(0), \gamma_h{}'(0)) = 1$. Moreover, the spherical wavefront at any time τ is the set $\{\gamma_F(\tau)\}$.

Proof. Given any point $p = (x, y)$ of M, since the conditions, such as the distribution of fuel, temperature, and etc., are smooth throughout M, these conditions are uniform across T_pM. Therefore, by Lemma 2.1, the spherical wavefront in T_pM is a translation of the rotated ellipse

$$Q(u, v) = (\frac{u\cos\alpha - v\sin\alpha}{a})^2 + (\frac{u\sin\alpha + v\cos\alpha}{b})^2 = 1,$$

in which a, b, and α are constant in T_pM and are determined from the experimental data. Therefore, in M, Q is given by

$$Q\big((x,y),(u,v)\big) = (\frac{u\cos\alpha(x,y) - v\sin\alpha(x,y)}{a(x,y)})^2 + (\frac{u\sin\alpha(x,y) + v\cos\alpha(x,y)}{b(x,y)})^2 = 1,$$

in which a, b, and α are smooth functions in M such that at each point $p = (x, y) \in M$ they correspond to some rotated ellipse in T_pM. In fact in T_pM one can write $Q = (u, v)[h_{ij}](u, v)^T$, where B^T is the transpose of the matrix B, and $h = \frac{1}{2}\mathtt{Hess}\, Q(u, v) = \frac{1}{2}[\frac{\partial^2 Q}{\partial u_i \partial u_j}]$. Hence the Riemannian metric in M is given by

$$h(x, y) = \frac{1}{2}\mathtt{Hess}\, Q\big((x,y),(u,v)\big) = \frac{1}{2}[\frac{\partial^2 Q}{\partial u_i \partial u_j}].$$

The rest of proof is deduced from Theorem 2.3. Because, once for W we have

$$\sum_{i,j,k=1}^{2} W_k \frac{\partial h_{ij}}{\partial x_k} = 0,$$

the conditions of Theorem 2.3 are satisfied and therefore this theorem can be applied. It is not difficult to see that the flow of W is $\varphi(t, q) = tW + q$, where $q \in M$. Hence, from item (i) of Theorem 2.3, we have $\gamma_F(t) = Wt + \gamma_h(t)$, such that $\gamma_h(0) = p$ and $h(\gamma_h{}'(0), \gamma_h{}'(0)) = 1$. Also from item (ii) of Theorem 2.3, the spherical wavefront at each time τ is the set $\{\gamma_F(\tau)\}$. □

4 EXAMPLE

In this example, we simulate a wildfire spreading in the *Boa Nova National Park* in Brazil. This park is the house of about 437 species of birds, an endangered bird species, and attracts many tourists. The wildfire is spreading west of the *slender antbirds* zone, and the wind blows toward the north with a velocity of $1/3$. From the experimental data we are given: $a = 1.3$, $b = 3.2$, and $\alpha = 2/3 - y$. We simulate the fire propagation model and focus on 1-finding the path through which fire will progress faster and 2- it will reach the location where a big group of *slender antbirds* nest.

Fig. 1 shows the park map and Figs. 2 (a) to (d) show the wavefronts, wave rays, and propagation models after 1, 5, 10, and 20 hours, respectively. The blue arrows display the wind vector, and by Δt, we mean the time between every two successive wavefronts. In Fig. 2 (a) to (d), the thick black path is the path via which the fire advances faster.

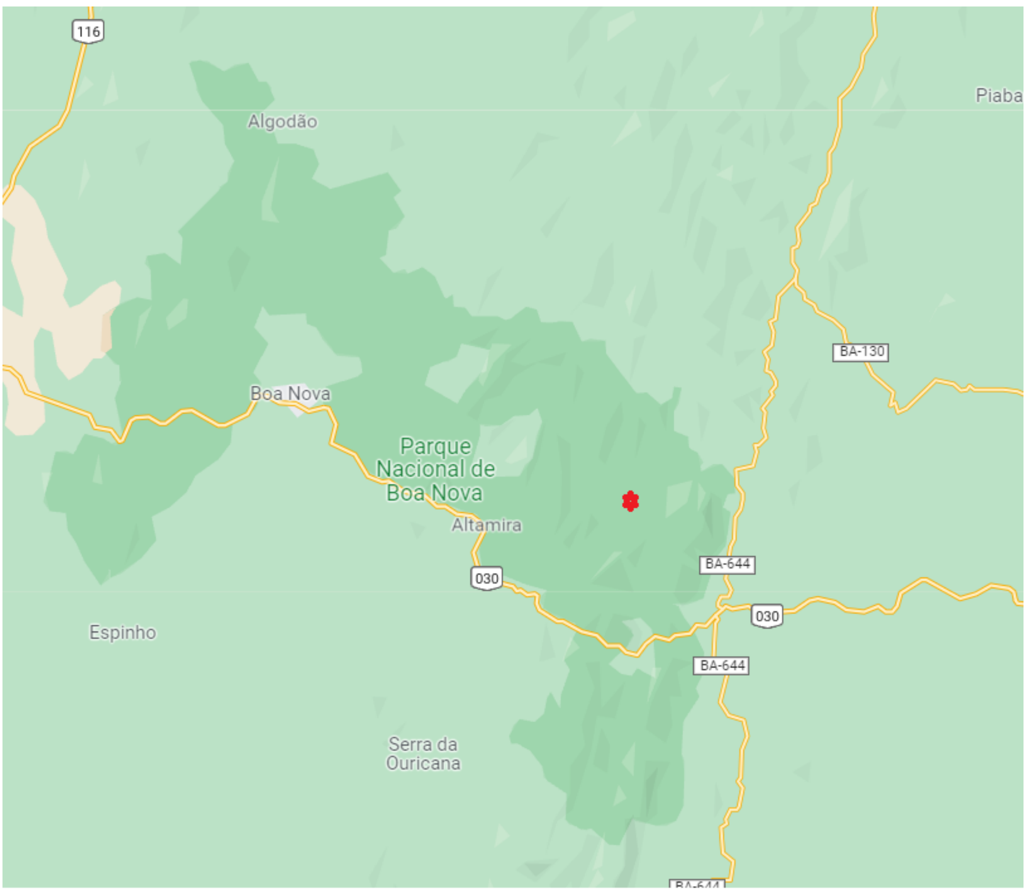
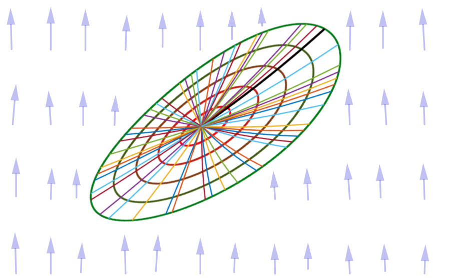
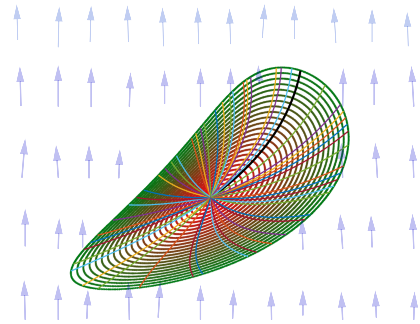


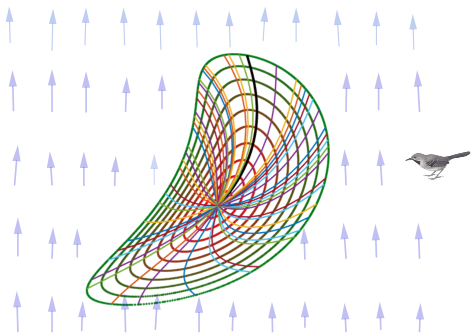
Figure 1: The map of Boa Nova national park with the initial point of fire marked with red color.



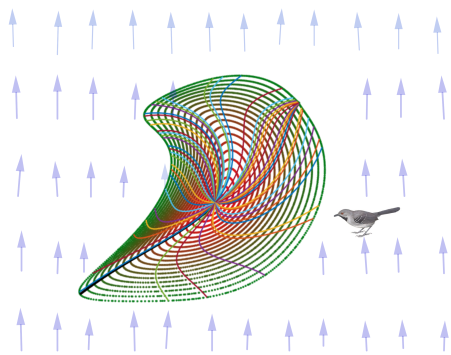
(a) Propagation after 1 hour with $\Delta t = 12$ min.



(b) Propagation after 5 hours with $\Delta t = 12$ min.



(c) Propagation after 10 hours with $\Delta t = 1$ hour.



(d) Propagation after 20 hours with $\Delta t = 1$ hour

Figure 2: The fire shape, wave rays and wavefronts for the wildfire propagation in Boa Nova national park.

According to Fig. 2 (a), during the first hour of spreading, the fire rays are straight lines, and the fire propagates in an ellipse shape. The fire has its fastest progress toward the northeast. After 6 hours, Fig.2 (b), the fire's shape has changed, and the fire tends to spread faster towards the north. However, it is still far from the *slender antbirds* zone. After 16 hours of spreading, Fig. 2 (c) says that the fire is still progressing toward the north faster and also is getting close to the *slender antbirds*. According to Fig. 2 (d), after 36 hours, the fire moves faster toward the southwest, and it will surround the *slender antbirds* via the orange path if it is not controlled soon.

5 CONCLUSION

In this work, for wildfire propagation in some flat land of zero slope, the equations of spherical wavefronts and wave rays were provided. Therefore, to provide the model of propagation, one can use the equations of wave rays igniting from the point from which the fire has started, and then find the wavefront at any time τ later, or find the equations of wave rays igniting from points belonging to some given wavefront and then apply the Huygens' principle.

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