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Laplace's and Poisson's Equations in a Semi-Disc under the Dirichlet-Neumann Mixed Boundary Condition

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ABSTRACT. In this work, the solution of Poisson's equation in a semi-disc under a Dirichlet boundary condition at the base and a Neumann boundary condition on the circumference is calculated. The solution is determined in terms of Green's function, which is calculated in two ways, by the method of images and by solving its equation. In the particular case of Laplace's equation, it is presented a second way to solve it, which uses separation of variables and a Fourier transform.

Keywords: Laplace, Poisson, semi-disc, Dirichlet, Neumann, Green's, images.

1 INTRODUCTION

In this work we solve the following problem formed with Poisson's equation in plane polar coordinates:

$$\begin{cases} \nabla^2 u(r,\theta) = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = h(r,\theta) \text{ with } r \in (0,b) \text{ and } \theta \in (0,\pi) \\ u(r,0) = 0 \ \text{ if } \ r \in [0,b] \\ u(r,\pi) = f(r) \ \text{ if } \ r \in (0,b] \\ \dfrac{\partial u}{\partial r}(b,\theta) = g(\theta) \ \text{ if } \ \theta \in (0,\pi) \end{cases} \tag{1.1}$$

where the functions f, g and h are continuous. It is a boundary value problem, whose domain Ω has the shape of the semi-disc shown in Figure 1, under a Dirichlet boundary condition at the base and a Neumann boundary condition on the circumference, for which we want a solution u that is continuous in $\Omega \cup \partial\Omega$.

We calculate the solution to this problem using Green's function. But when $h(r,\theta) \equiv 0$, that is, in the particular case of Laplace's equation, we also calculate the solution using separation of variables combined with a Fourier transform.

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We determine Green's function in two ways, first by using the method of images and then by solving its partial differential equation. This second way is accomplished by a reduction to a one-dimensional Green's function, being implemented the two possible reductions: to the one-dimensional Green's function dependent on the *radial* variable and to the one dependent on the *angular* variable.

In addition to presenting these various methods of calculating the desired solution, it is also our aim to show that the different expressions that are obtained for it represent the same solution. We achieve this goal by calculating sums of infinite series and performing complicated definite integrals to write all these expressions in exactly the same form.

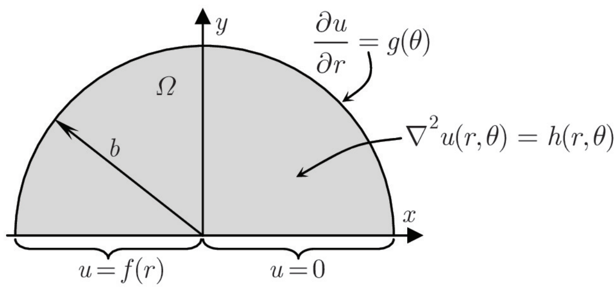

Figure 1: The problem that is solved in this work.

Section 2 contains the calculation of the solution to the problem in the case of Laplace's equation. Section 3 presents the calculation of the solution to the problem with Poisson's equation employing the Green's function determined by the method of images. Section 4 describes the determination of Green's function by solving its partial differential equation. Section 5 presents the final conclusions.

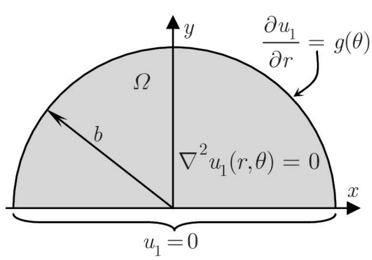

Figure 2: The problem for $u_1(r,\theta)$.

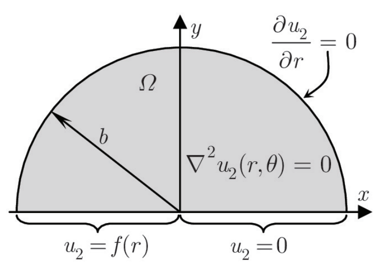

Figure 3: The problem for $u_2(r,\theta)$.

2 CALCULATION OF THE SOLUTION TO THE PROBLEM IN THE PARTICULAR CASE OF LAPLACE'S EQUATION

We can write the solution of the problem defined in (1.1) when $h(r,\theta) \equiv 0$ (Laplace's equation) as the superposition of the following two particular solutions of it:

$$u(r,\theta) = u_1(r,\theta) + u_2(r,\theta) \ ,$$

where $u_1(r,\theta)$ is the solution when $f(r) \equiv 0$, that is, when the boundary condition at the base is homogeneous (cf. Figure 2), and $u_2(r,\theta)$ is the solution when $g(\theta) \equiv 0$, that is, when the boundary condition on the circumference is homogeneous (cf. Figure 3). We can calculate $u_1(r,\theta)$ by separation of variables and $u_2(r,\theta)$ by a Fourier transform, as we see next.

2.1 Calculation of $u_1(r,\theta)$ by separation of variables

Substituting $u_1(r,\theta) = R(r)\Theta(\theta)$ into $\nabla^2 u_1 = 0$ and separating variables as usual, by means of a separation constant λ, we obtain

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \overset{\times \frac{r^2}{R\Theta}}{\Longrightarrow} \quad \overbrace{\frac{r^2R'' + rR'}{R}}^{\lambda} + \overbrace{\frac{\Theta''}{\Theta}}^{-\lambda} = 0$$

$$\Rightarrow \quad \begin{cases} \Theta'' + \lambda\Theta(\theta) = 0 \\ r^2R'' + rR' - \lambda R(r) = 0 \ . \end{cases} \tag{2.1}$$

From the homogeneous boundary conditions, we get

$$u_1(r,\theta) = R(r)\Theta(\theta) = 0 \ \text{ for } \ \theta = 0 \ \text{ and } \ \theta = \pi \quad \Rightarrow \quad \Theta(0) = \Theta(\pi) = 0 \ .$$

These conditions and the differential equation for $\Theta(\theta)$ in (2.1) form a boundary-value problem with well-known eigenvalues and eigenfunctions {cf. Ref. [12], sec. 5.2, Example 2}:

$$\begin{cases} \Theta'' + \lambda\Theta(\theta) = 0, \ \theta \in (0,\pi) \\ \Theta(0) = \Theta(\pi) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda_n = n^2 \ (n = 1,2,3\cdots) \\ \Theta_n(\theta) = \sin n\theta \end{cases} \ . \tag{2.2}$$

We now solve the differential equation for $R(r)$ in (2.1) with these eigenvalues of λ {cf. Ref. [12], sec. 4.7, eq. (2)}:

$$r^2R_n'' + rR_n' - n^2R_n(r) = 0 \quad \Rightarrow \quad R_n(r) = C_nr^n + D_n/r^n \ . \tag{2.3}$$

After setting $D_n = 0$ to avoid infinity at $r = 0$, we form the most general solution that is possible so far:

$$u_1(r,\theta) = \sum_{n=1}^{\infty} R_n(r)\Theta_n(\theta) = \sum_{n=1}^{\infty} C_nr^n \sin n\theta \ . \tag{2.4}$$

To calculate the constants C_n, we impose the boundary condition on the circumference; this produces a sine Fourier series of known coefficients from which those constants are determined:

$$\frac{\partial u_1}{\partial r}(b,\theta) = \sum_{n=1}^{\infty} nC_n b^{n-1} \sin n\theta = g(\theta) \;\Rightarrow\; nC_n b^{n-1} = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta\, d\theta\,. \tag{2.5}$$

With this result the present calculation is complete: $u_1(r,\theta)$ is given by (2.4) with C_n taken from (2.5).

But we do not finish here, because, fortunately, something interesting happens in the expression obtained for $u_1(r,\theta)$: the summation in (2.4) can be carried out. To accomplish this, we substitute the expression of C_n given by (2.5) (being necessary, before the substitution, to change the notation of the integration variable, say from θ to θ') into the expression of $u_1(r,\theta)$ given by (2.4), obtaining

$$\begin{aligned} u_1(r,\theta) &= \sum_{n=1}^{\infty}\left[\frac{2}{\pi n b^{n-1}}\int_0^{\pi} g(\theta')\sin n\theta'\, d\theta'\right] r^n \sin n\theta \\ &= \frac{b}{\pi}\int_0^{\pi}\Big[\sum_{n=1}^{\infty}\frac{r^n 2\sin n\theta' \sin n\theta}{nb^n}\Big] g(\theta')\, d\theta'\,. \end{aligned} \tag{2.6}$$

The summation between brackets above, by using the identity $2\sin n\theta' \sin n\theta = \cos n(\theta'-\theta) - \cos n(\theta'+\theta)$, can be written as follows:

$$\sum_{n=1}^{\infty}\frac{2}{n}\left(\frac{r}{b}\right)^n \sin n\theta' \sin n\theta = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n \cos n(\theta'-\theta) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n \cos n(\theta'+\theta)\,,$$

where in the right-hand side we recognize two series of the form of the series $S_+(p,\varphi)$ defined in equation (A.1) of the Appendix, both with $p = r/b$, but the first one with $\varphi = \theta' - \theta$ and the second one with $\varphi = \theta' + \theta$. Thus, using the formula for this series given in that equation, we get

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{2}{n}\left(\frac{r}{b}\right)^n \sin n\theta' \sin n\theta \;=\; S_+(r/b\,,\,\theta'-\theta) - S_+(r/b\,,\,\theta'+\theta) \\ &= -\frac{1}{2}\ln\left[1+\frac{r^2}{b^2}-2\frac{r}{b}\cos(\theta'-\theta)\right] + \frac{1}{2}\ln\left[1+\frac{r^2}{b^2}-2\frac{r}{b}\cos(\theta'+\theta)\right] \\ &= \frac{1}{2}\ln\frac{r^2+b^2-2br\cos(\theta'+\theta)}{r^2+b^2-2br\cos(\theta'-\theta)}\;. \end{aligned} \tag{2.7}$$

Therefore, using (2.7) in (2.6), we finally obtain

$$u_1(r,\theta) = \frac{b}{2\pi}\int_0^{\pi} \ln\frac{r^2+b^2-2br\cos(\theta'+\theta)}{r^2+b^2-2br\cos(\theta'-\theta)}\, g(\theta')\, d\theta'\;. \tag{2.8}$$

2.2 Calculation of $u_2(r,\theta)$ by a Fourier transform

Consider the new variable ρ related to r as follows:

$$r = b e^{-\rho} \in [0,b] \quad\Leftrightarrow\quad \rho = -\ln(r/b) \in [0,\infty)\;. \tag{2.9}$$

Using the chain rule, we have

$$u_2(r,\theta) = u_2(be^{-\rho},\theta) \equiv U_2(\rho,\theta) \quad \Rightarrow \quad \begin{cases} r\dfrac{\partial u_2}{\partial r}(r,\theta) = -\dfrac{\partial U_2}{\partial \rho}(\rho,\theta) \\ r^2\dfrac{\partial^2 u_2}{\partial r^2} = \dfrac{\partial^2 U_2}{\partial \rho^2} + \dfrac{\partial U_2}{\partial \rho} \quad . \end{cases}$$

Consequently, Laplace's equation

$$\nabla^2 u_2(r,\theta) = 0\,, \text{ or } \; r^2\frac{\partial^2 u_2}{\partial r^2} + r\frac{\partial u_2}{\partial r} + \frac{\partial^2 u_2}{\partial \theta^2} = 0 \;,$$

takes the simpler form

$$\frac{\partial^2 U_2}{\partial \rho^2} + \frac{\partial^2 U_2}{\partial \theta^2} = 0 \;,$$

and the homogeneous boundary conditions become

$$U_2(\rho,0) = u_2(r,0) = 0 \quad \text{and} \quad \frac{\partial U_2}{\partial \rho}(\rho,\theta)\Big|_{\rho=0} = \Big[-r\frac{\partial u_2}{\partial r}(r,\theta)\Big]_{r=b} = 0 \;,$$

that is, with the variable ρ instead of r the problem we are to solve to determine u_2 becomes

$$\begin{cases} \dfrac{\partial^2 U_2}{\partial \rho^2} + \dfrac{\partial^2 U_2}{\partial \theta^2} = 0\;, \text{ with } \rho \in (0,\infty) \text{ and } \theta \in (0,\pi) \\ U_2(\rho,0) = 0 \text{ if } \rho \in [0,\infty)\;, \; \dfrac{\partial U_2}{\partial \rho}(0,\theta) = 0 \text{ if } \theta \in (0,\pi) \\ U_2(\rho,\pi) = f(r) = f(be^{-\rho}) \equiv F(\rho) \text{ if } \rho \in [0,\infty) \quad . \end{cases} \tag{2.10}$$

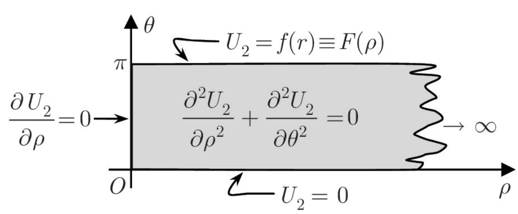

Figure 4: The problem to be solved to determine $U_2(\rho,\theta) = u_2(r,\theta)$.

This problem is depicted in Figura 4.

We see that, in the plane of ρ and θ, the problem domain takes the shape of a semi-infinite slab; this fact and the homogeneous Neumann condition on the boundary at $\rho = 0$ justifies the use of the following transform:

$$\mathcal{F}_c\{U_2(\rho,\theta)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty U_2(\rho,\theta)\cos k\rho\, d\rho \;\equiv\; \bar{U}_2(k,\theta) \;;$$

it is a cosine Fourier transform. Using it to transform the partial differential equation in (2.10) and taking into account the boundary condition at $\rho = 0$, we obtain

$$-k^2\bar{U}_2(k,\theta) - \sqrt{\frac{2}{\pi}}\,\overbrace{\frac{\partial U_2}{\partial \rho}(0,\theta)}^{0} + \frac{d^2\bar{U}_2}{d\theta^2} = 0$$

$$\Rightarrow \quad \bar{U}_2(k,\theta) = c_1\cosh k\theta + c_2\sinh k\theta \ . \tag{2.11}$$

The constants c_1 and c_2 are determined from the transforms of the boundary conditions at $\theta = 0$ and $\theta = \pi$:

$$U_2(\rho,0) = 0 \quad \overset{\mathcal{F}_c}{\Longrightarrow} \quad \bar{U}_2(k,0) = c_1 = 0 \ ,$$

$$U_2(\rho,\pi) = F(\rho) \quad \overset{\mathcal{F}_c}{\Longrightarrow} \quad \bar{U}_2(k,\pi) = c_2\sinh k\pi = \mathcal{F}_c\{F(\rho)\} \ \Rightarrow \ c_2 = \frac{\mathcal{F}_c\{F(\rho)\}}{\sinh k\pi} \ .$$

Equation (2.11) then becomes

$$\bar{U}_2(k,\theta) = \frac{\mathcal{F}_c\{F(\rho)\}}{\sinh k\pi}\sinh k\theta = \sqrt{\frac{2}{\pi}}\,\frac{\sinh k\theta}{\sinh k\pi}\int_0^\infty F(\rho')\cos k\rho'\,d\rho' \ ,$$

and the next step is to write the inverse transform of the above result:

$$U_2(\rho,\theta) = \mathcal{F}_c^{-1}\{\bar{U}_2(k,\theta)\} = \mathcal{F}_c^{-1}\left\{\sqrt{\frac{2}{\pi}}\,\frac{\sinh k\theta}{\sinh k\pi}\int_0^\infty F(\rho')\cos k\rho'\,d\rho'\right\}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{\sqrt{\frac{2}{\pi}}\,\frac{\sinh k\theta}{\sinh k\pi}\int_0^\infty F(\rho')\cos k\rho'\,d\rho'\right\}\cos k\rho\,dk \ .$$

With this result we have completely determined $U_2(\rho,\theta) = u_2(r,\theta)$.

As in the calculation of $u_1(r,\theta)$, we do not finish here, because we can improve the above result: we can carry out the integration with respect to k. To do this, we first change the order of the two integrations to write

$$U_2(\rho,\theta) = \frac{1}{\pi}\int_0^\infty \Bigg(\underbrace{\int_0^\infty \frac{\sinh k\theta}{\sinh k\pi}\,2\cos k\rho'\cos k\rho\,dk}_{I}\Bigg)F(\rho')\,d\rho' \ , \tag{2.12}$$

where we denote the integral we want to calculate by I.

Then, by using the trigonometric identity $2\cos k\rho'\cos k\rho = \cos k(\rho'-\rho) + \cos k(\rho'+\rho)$, we write I as

$$I = \int_0^\infty \frac{\sinh k\theta}{\sinh k\pi}\cos k(\rho'-\rho)\,dk + \int_0^\infty \frac{\sinh k\theta}{\sinh k\pi}\cos k(\rho'+\rho)\,dk \ ,$$

that is, involving integrals that have already been calculated, given by the Formula 3.981-5 in Ref. [6], which we transcribe adapted to our problem:

$$\int_0^\infty \frac{\sinh \beta x}{\sinh \gamma x}\cos ax\,dx = \frac{\pi}{2\gamma}\cdot\frac{\sin(\pi\beta/\gamma)}{\cosh(\pi a/\gamma) + \cos(\pi\beta/\gamma)} \quad (0 \le \beta < \gamma) \ .$$

Using it, we obtain

$$\begin{aligned} I &= \frac{(1/2)\sin\theta}{\cosh(\rho'-\rho)+\cos\theta} + \frac{(1/2)\sin\theta}{\cosh(\rho'+\rho)+\cos\theta} \\ &= \frac{(1/2)\sin\theta}{\dfrac{r'^2+r^2}{2rr'}+\cos\theta} + \frac{(1/2)\sin\theta}{\dfrac{b^2+(rr'/b)^2}{2rr'}+\cos\theta} \\ &= \left[\frac{1}{r^2+r'^2+2rr'\cos\theta} + \frac{1}{(rr'/b)^2+b^2+2rr'\cos\theta}\right] rr'\sin\theta \quad (\theta<\pi)\ , \end{aligned}$$

where we have already eliminated ρ and ρ', thus returning to the variables r and r'.

Finally, substituting this result for I into (2.12), remembering that $U_2(\rho,\theta) = u_2(r,\theta)$, and taking into account that $\int_0^\infty F(\rho')\,d\rho' = \int_b^0 f(r')\,(-dr'/r')$, we complete the calculation of u_2 (for $\theta < \pi$):

$$u_2(r,\theta) = \frac{r\sin\theta}{\pi}\int_0^b \left[\frac{1}{r^2+r'^2+2rr'\cos\theta} + \frac{1}{(rr'/b)^2+b^2+2rr'\cos\theta}\right] f(r')\,dr'\ . \qquad (2.13)$$

3 CALCULATION OF THE SOLUTION TO THE PROBLEM WITH POISSON'S EQUATION

3.1 The solution in terms of Green's function

The value of the solution u of the problem in (1.1) at a point $\vec{r}$ of the domain Ω can be expressed by Green's representation theorem {cf. Ref. [7], eq. (1.42) ; Ref. [4], eq. (5.0.13) ; and Ref. [10], ch. IV, §4, eq. (3)} :

$$u(\vec{r}) = -\frac{1}{2\pi}\iint_\Omega G(\vec{r}\,|\vec{r}\,')h(\vec{r}\,')\,dA' + \int_{\partial\Omega}\left[G(\vec{r}\,|\vec{r}\,')\frac{\partial u}{\partial n'}(\vec{r}\,') - \frac{\partial G}{\partial n'}(\vec{r}\,|\vec{r}\,')\,u(\vec{r}\,')\right]ds'\ , \qquad (3.1)$$

where the line integral along the boundary $\partial\Omega$ is traveled in the conventional positive direction. The Green's function $G(\vec{r}\,|\vec{r}\,')$ of our bidimensional problem is the sum of the harmonic function $\ln\left[1/|\vec{r}\,'-\vec{r}\,|\right]$ $(\vec{r}\,'\neq\vec{r}\,)$ {cf. Ref. [11], sec. VII.3, Example 3.1} and an arbitrary harmonic function $v(\vec{r}\,)$ in Ω {cf. Ref. [11], sec. VII.9, Definition 9.2 ; and Ref. [10], p. 361}, that is,

$$\begin{cases} G(\vec{r}\,|\vec{r}\,') = \ln\dfrac{1}{|\vec{r}\,'-\vec{r}\,|} + v(\vec{r}\,')\ , \text{ with } \vec{r} \text{ and } \vec{r}\,' \text{ in } \bar{\Omega}\equiv\Omega\cup\partial\Omega\ , & (3.2)\\ \text{where } \nabla'^2 v(\vec{r}\,') = 0 \text{ if } \vec{r}\,'\in\Omega\ , & (3.3)\end{cases}$$

where the prime placed as a superscript in ∇' indicates that the derivatives in this operator are taken with respect to the coordinates of the position vector $\vec{r}\,'$.

Let $\partial\Omega = \partial\Omega_1\cup\partial\Omega_2$, where $\partial\Omega_1$ and $\partial\Omega_2$ denote, as Figure 5 shows, the base and the circumference of the semi-disc Ω, respectively. Under the conditions

$$\text{(i) } G(\vec{r}\,|\vec{r}\,') = 0 \text{ if } \vec{r}\,'\in\partial\Omega_1 \qquad \text{and} \qquad \text{(ii) } \frac{\partial G}{\partial n'}(\vec{r}\,|\vec{r}\,') = 0 \text{ se } \vec{r}\,'\in\partial\Omega_2\ , \qquad (3.4)$$

which eliminates the unknown values of $\partial u/\partial n'$ on $\partial\Omega_1$ and of u on $\partial\Omega_2$ in (3.1), this equation furnishes the following expression for the solution of (1.1):

$$u(\vec{r}) \;=\; -\frac{1}{2\pi}\iint_{\Omega} G(\vec{r}\,|\vec{r}\,')h(\vec{r}\,')\,dA' \\ \underbrace{-\frac{1}{2\pi}\int_{\partial\Omega_1}\frac{\partial G}{\partial n'}(\vec{r}\,|\vec{r}\,')\,u(\vec{r}\,')\,ds'}_{u_2(r,\theta)} + \underbrace{\frac{1}{2\pi}\int_{\partial\Omega_2} G(\vec{r}\,|\vec{r}\,')\frac{\partial u}{\partial n'}(\vec{r}\,')\,ds'}_{u_1(r,\theta)} \,. \tag{3.5}$$

We indicate above one term by $u_1(r,\theta)$ and the other by $u_2(r,\theta)$ because, as we will show, they are exactly those in (2.8) and (2.13). Therefore, as expected, when $h(\vec{r}\,')\equiv 0$, the above equation becomes the Laplace's equation solution calculated in Section 2.

At the stage of calculating the Green's function to be used in equation (3.5), $\vec{r}$ is a parameter, being the fixed point where the solution is calculated, and $\vec{r}\,'$ is the variable, being the vector variable of integration over Ω as well as along the boundary $\partial\Omega$. In view of (3.2), the two conditions in (3.4) are also conditions for $v(\vec{r}\,')$, and we thus have a well-posed problem for this function, and a corresponding one for Green's function. We do not solve any of these problems in this section[1]; here we determine these two functions by using the method of images {cf. Ref. [11], sec. VII.13; and Ref. [9], sec. 10.2}.

3.2 Green's function determination by the method of images

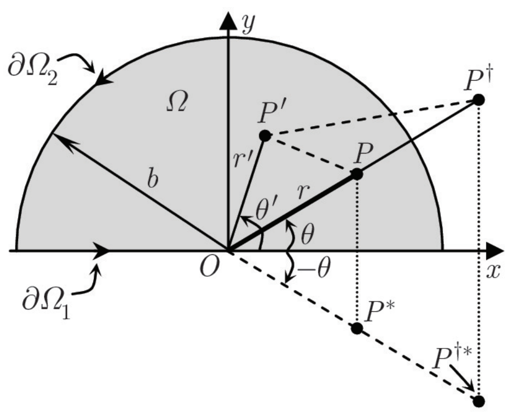

Figure 5: The images necessary to form Green's function.

As our problem involves both the Dirichlet and Neumann conditions, here we combine what the method of images prescribes in each of these two cases of boundary conditions. In the geometry of our problem, the method of images prescribes three imagens of the point P at the position $\vec{r}$, those at the three points P^*, $P^\dagger$, and $P^{\dagger *}$ shown in Figure 5, which, in view of the boundary conditions, must be as follows: The image P^* of P is obtained by reflection with respect to

[1]This is done in Section 4, where we calculate $G(\vec{r}\,|\vec{r}\,')$ by solving its well-posed problem.

the x-axis in order to satisfy the Dirichlet boundary condition; the image $P^\dagger$ of P is obtained by inversion with respect to the circumference [11, sec. VII.3] in order to satisfy the Neumann boundary condition; and $P^{\dagger *}$ is an image of images, obtained either by reflection of $P^\dagger$ with respect to the x-axis or by inversion of P^* with respect to the (complete) circumference, thus contributing for both the Dirichlet and Neumann conditions to be satisfied. So we have four object-image pairs: PP^* and $P^\dagger P^{\dagger *}$, which guarantees the Dirichlet condition on the x-axis, as well as $PP^\dagger$ and $P^* P^{\dagger *}$, which guarantees the Neumann condition on the semicircumference.

By writing the *two* logarithmic terms prescribed to each object-image *pair* (according to the rule that the signs of these two terms must be different or equal depending respectively on whether a Dirichlet or Neumann condition is to be satisfied), we obtain the following Green's function:

$$G(\vec{r}\,|\vec{r}\,') = \ln\frac{1}{|\vec{r}\,'-\vec{r}\,|} \underbrace{-\ln\frac{1}{|\vec{r}\,'-\vec{r}\,^*|}+\ln\frac{b/r}{|\vec{r}\,'-\vec{r}\,^\dagger|}-\ln\frac{b/r}{|\vec{r}\,'-\vec{r}\,^{\dagger *}|}}_{v(\vec{r}\,')} \ , \tag{3.6}$$

where $\vec{r}\,^*$, $\vec{r}\,^\dagger$, and $\vec{r}\,^{\dagger *}$ are the position vectors of P^*, $P^\dagger$, and $P^{\dagger *}$. The plane polar coordinates of the position vectors in (3.6) are as follows:

$$\vec{r}\,(r,\theta)\ ,\quad \vec{r}\,'(r',\theta')\ ,\quad \vec{r}\,^*(r,-\theta)\ ,\quad \vec{r}\,^\dagger(b^2/r,\theta)\ ,\quad \vec{r}\,^{\dagger *}(b^2/r,-\theta)\ .$$

In equation (3.6), each magnitude of a difference of vectors is a distance from P' to one of the other points considered. We list below these distances as functions of the plane polar coordinates of the vector variables $\vec{r}$ and $\vec{r}\,'$ of $G(\vec{r}\,|\vec{r}\,')$ (they can be calculated either analytically, by using the definition of magnitude of a vector, or geometrically, by the law of cosines):

$$\begin{cases} |\vec{r}\,'-\vec{r}\,| &= \sqrt{r^2+r'^2-2rr'\cos(\theta'-\theta)}\ , \\ |\vec{r}\,'-\vec{r}\,^\dagger| &= (b/r)\sqrt{(rr'/b)^2+b^2-2rr'\cos(\theta'-\theta)}\ , \\ |\vec{r}\,'-\vec{r}\,^*| &= \sqrt{r^2+r'^2-2rr'\cos(\theta'+\theta)}\ , \\ |\vec{r}\,'-\vec{r}\,^{\dagger *}| &= (b/r)\sqrt{(rr'/b)^2+b^2-2rr'\cos(\theta'+\theta)}\ . \end{cases} \tag{3.7}$$

Substitution of (3.7) into (3.6) yields

$$\begin{aligned} G(\vec{r}\,|\vec{r}\,') &= G(r,\theta|r',\theta') \\ &= -\frac{1}{2}\ln\big[r^2+r'^2-2rr'\cos(\theta'-\theta)\big] \quad +\frac{1}{2}\ln\big[r^2+r'^2-2rr'\cos(\theta'+\theta)\big] \\ &\quad -\frac{1}{2}\ln\Big[\big(\tfrac{rr'}{b}\big)^2+b^2-2rr'\cos(\theta'-\theta)\Big]+\frac{1}{2}\ln\Big[\big(\tfrac{rr'}{b}\big)^2+b^2-2rr'\cos(\theta'+\theta)\Big] \qquad (3.8a) \\ &= \frac{1}{2}\ln\frac{(rr'/b)^2+b^2-2rr'\cos(\theta'+\theta)}{(rr'/b)^2+b^2-2rr'\cos(\theta'-\theta)}+\frac{1}{2}\ln\frac{r^2+r'^2-2rr'\cos(\theta'+\theta)}{r^2+r'^2-2rr'\cos(\theta'-\theta)} \qquad (3.8b) \\ &= \frac{1}{2}\ln\frac{\Big[\big(\tfrac{rr'}{b}\big)^2+b^2-2rr'\cos(\theta'+\theta)\Big]\big[r^2+r'^2-2rr'\cos(\theta'+\theta)\big]}{\Big[\big(\tfrac{rr'}{b}\big)^2+b^2-2rr'\cos(\theta'-\theta)\Big]\big[r^2+r'^2-2rr'\cos(\theta'-\theta)\big]}\ , \qquad (3.8c) \end{aligned}$$

where we provide different but equivalent expressions because each is the most appropriate one to use for certain operations.

That (3.6), or (3.8), provides the correct Green's function is easily verified. Notice that the function $v(\vec{r}\,')$ indicated in (3.6) is indeed harmonic in Ω, since $\vec{r}^{\,*}$, $\vec{r}^{\,\dagger}$ and $\vec{r}^{\,\dagger *}$ are not points of this domain. That (3.6) also satisfies condition (i) in (3.4) is as a consequence of the equalities $|\vec{r}\,'-\vec{r}|=|\vec{r}\,'-\vec{r}^{\,*}|$ and $|\vec{r}\,'-\vec{r}^{\,\dagger}|=|\vec{r}\,'-\vec{r}^{\,\dagger *}|$ when $\vec{r}\,'\in\partial\Omega$, what becomes evident by imagining, in Figure 5, that $P'\in\partial\Omega_1$. To show that condition (ii) on (3.4) is satisfied, we calculate the normal derivative of Green's function using (3.8a) and verify that it does vanishes on the circumference:

$$
\begin{aligned}
\frac{\partial G}{\partial n'}(\vec{r}|\vec{r}\,')\bigg|_{\vec{r}\,'\in\partial\Omega_2} &= \frac{\partial G}{\partial r'}\bigg|_{r'=b} \\
&= -\frac{b-r\cos(\theta'-\theta)}{r^2+b^2-2br\cos(\theta'-\theta)}+\frac{b-r\cos(\theta'-\theta)}{r^2+b^2-2br\cos(\theta'+\theta)} \\
&\quad -\frac{r^2/b-br\cos(\theta'-\theta)}{r^2+b^2-2br\cos(\theta'-\theta)}+\frac{r^2/b-br\cos(\theta'-\theta)}{r^2+b^2-2br\cos(\theta'+\theta)} = 0\,\checkmark
\end{aligned}
$$

It is thus confirmed that (3.8) is the correct Green's function.

3.3 Solution calculation

The solution of the problem in (1.1) in terms of Green's function is given by (3.5). Let us calculate the first term in this equation; using (3.8c), it becomes

$$
\begin{aligned}
&-\frac{1}{2\pi}\iint_\Omega G(\vec{r}|\vec{r}\,')h(\vec{r}\,')\,dA' = \\
&\frac{-1}{4\pi}\int_0^\pi\!\!\int_0^b \ln\frac{\left[\left(\frac{rr'}{b}\right)^2+b^2-2rr'\cos(\theta'+\theta)\right]\left[r^2+r'^2-2rr'\cos(\theta'+\theta)\right]}{\left[\left(\frac{rr'}{b}\right)^2+b^2-2rr'\cos(\theta'-\theta)\right]\left[r^2+r'^2-2rr'\cos(\theta'-\theta)\right]}\,h(r',\theta')\,r'dr'd\theta'. \qquad (3.9)
\end{aligned}
$$

Let us again use (3.8c) to calculate the term indicated by $u_1(r,\theta)$ in (3.5). The integrand in this term can be developed as follows:

$$
\begin{aligned}
&\left[G(\vec{r}|\vec{r}\,')\frac{\partial u}{\partial n'}(\vec{r}\,')\,ds'\right]_{\vec{r}\,'\in\partial\Omega_2} = \\
&=\frac{1}{2}\ln\frac{\left[\left(\frac{rr'}{b}\right)^2+b^2-2rr'\cos(\theta'+\theta)\right]\left[r^2+r'^2-2rr'\cos(\theta'+\theta)\right]}{\left[\left(\frac{rr'}{b}\right)^2+b^2-2rr'\cos(\theta'-\theta)\right]\left[r^2+r'^2-2rr'\cos(\theta'-\theta)\right]}\Bigg|_{r'=b} g(\theta')\,b\,d\theta' \\
&=b\ln\left[\frac{r^2+b^2-2br\cos(\theta'+\theta)}{r^2+b^2-2br\cos(\theta'-\theta)}\right]g(\theta')\,d\theta'.
\end{aligned}
$$

By substituting this result into the expression for $u_1(r,\theta)$ indicated in (3.5), we obtain exactly (2.8), the part of the solution already obtained by separation of variables:

$$
u_1(r,\theta)=\frac{b}{2\pi}\int_0^\pi \ln\left[\frac{r^2+b^2-2br\cos(\theta'+\theta)}{r^2+b^2-2br\cos(\theta'-\theta)}\right]g(\theta')\,d\theta'\;. \qquad (3.10)
$$

To complete the solution calculation, let us calculate the term indicated by $u_2(r,\theta)$ in (3.5). The integrand in this term, using (3.8a), can be developed as follows:

$$\begin{aligned}\left[\frac{\partial G}{\partial n'}(\vec{r}\,|\vec{r}\,')\,u(\vec{r}\,')\,ds'\right]_{\vec{r}\,'\in\partial\Omega_1} &= \left[\frac{1}{r'}\frac{\partial G}{\partial\theta'}(r,\theta|r',\theta')\,u(r',\theta')\,(-dr')\right]_{\theta'=\pi}\\ = -\frac{f(r')\,dr'}{r'}\left[\frac{\partial G}{\partial\theta'}(r,\theta|r',\theta')\right]_{\theta'=\pi}&\\ = \frac{f(r')\,dr'}{r'}\left[\frac{2r'r\sin\theta}{r^2+r'^2+2r'r\cos\theta}+\frac{2r'r\sin\theta}{(r'r/b)^2+b^2+2r'r\cos\theta}\right]\ ,&\end{aligned}$$

whose substitution into (3.5) yields

$$\begin{aligned}u_2(r,\theta) &= -\frac{1}{2\pi}\int_b^0\frac{f(r')\,dr'}{r'}\left[\frac{2r'r\sin\theta}{r^2+r'^2+2r'r\cos\theta}+\frac{2r'r\sin\theta}{(r'r/b)^2+b^2+2r'r\cos\theta}\right]\\ &= \frac{r\sin\theta}{\pi}\int_0^b\left[\frac{1}{r^2+r'^2+2r'r\cos\theta}+\frac{1}{(r'r/b)^2+b^2+2r'r\cos\theta}\right]f(r')\,dr'\ .\end{aligned} \tag{3.11}$$

This result is exactly (2.13), the part of the solution already obtained by a cosine Fourier transform.

The solution of the problem in (1.1) is then given by the sum of the expressions in (3.9), (3.10) and (3.11).

4 GREEN'S FUNCTION CALCULATION BY SOLVING THE WELL-POSED PROBLEM FOR IT

In this section we present another method of calculating Green's function; by solving its differential equation,

$$\begin{cases}\nabla'^2 G(r,\theta|r',\theta') = \dfrac{\partial^2 G}{\partial r'^2}+\dfrac{1}{r'}\dfrac{\partial G}{\partial r'}+\dfrac{1}{r'^2}\dfrac{\partial^2 G}{\partial\theta'^2} = -\dfrac{2\pi}{r'}\,\delta(r'-r)\,\delta(\theta'-\theta)\ ,\\ \text{with } r'\in(0,b) \text{ and } \theta'\in(0,\pi)\ ,\end{cases} \tag{4.1}$$

subjected to the boundary conditions given by (3.4):

$$\begin{cases}G(r,\theta|r',\theta') = 0 \ \text{ if } \ \theta'=0 \text{ or } \theta'=\pi \ \ \forall\, r'\in[0,b]\ , & (4.2\text{a})\\ \dfrac{\partial G}{\partial r'}(r,\theta|r',\theta') = 0 \ \text{ if } \ r'=b \ \ \forall\,\theta'\in(0,\pi)\ . & (4.2\text{b})\end{cases}$$

In this method, Green's function is considered a generalized function, involving Dirac's delta function in its formulation. Its differential equation in (4.1) is derived by applying the operator ∇'^2 to each term in equation (3.2), using the identity $\nabla'^2\left[\ln\left(1/|\vec{r}\,'-\vec{r}\,|\right)\right] = -2\pi\,\delta(\vec{r}\,'-\vec{r}\,)$ [2, sec. 4.4] as well as equation (3.3), and expressing the resulting equation in plane polar coordinates.

We will solve the above partial differential equation for $G(r,\theta|r',\theta')$ by reducing it to an ordinary differential equation in two ways: first forming a problem for a one-dimensional Green's function that depends on the radial variable r', then another that depends on the angular variable θ'

{cf. Ref. [3], sec. 12.4, p. 521 *et seq.*; and Ref. [5], sec. 5.7.2}. These two ways of implementing the reduction is presented in the following two subsections.

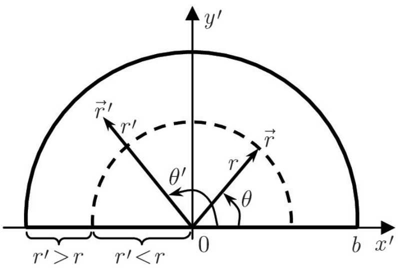

Figure 6: *Circular* division of Ω in two subdomains.

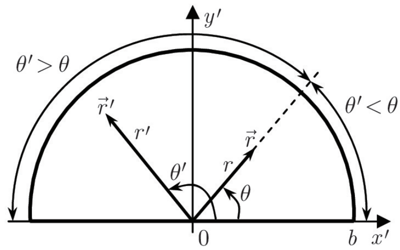

Figure 7: *Sectorial* division of Ω in two subdomains.

4.1 Reduction to a one-dimensional Green's function dependent on the radial variable

Consider the domain Ω divided in a inner subdomain where $r' < r$ and a outer one where $r' > r$; this *circular* division of the semi-disc is shown in Figure 6. Since Green's function equation in (4.1) is homogeneous for $r' \neq r$, we can solve that partial differential equation in each subdomain by separation of variables, that is, by substituting

$$G(r,\theta|r',\theta') = R(r')\Theta(\theta') \tag{4.3}$$

into (4.1), obtaining

$$\frac{r'^2 d^2R/dr'^2 + r' dR/dr'}{R} + \underbrace{\frac{d^2\Theta/d\theta'^2}{\Theta}}_{-\lambda} = 0 \quad (r' \neq r) \ .$$

The ordinary differential equation for $\Theta(\theta')$ which is obtained by using the separation constant λ as indicated above together with the boundary conditions for $\Theta(\theta')$ which are derived from (4.2a) using (4.3) form the same eigenvalue problem in (2.2). Therefore, we can use the eigenfunctions $\Theta_n(\theta')$ to replace (4.3) by the more general expression

$$G(r,\theta|r',\theta') = \sum_{n=1}^{\infty} R_n(r') \sin n\theta' \ . \tag{4.4}$$

To determine $R_n(r')$, we substitute (4.4) into Green's function equation (4.1), obtaining

$$\sum_{n=1}^{\infty} \left[r'^2 \frac{d^2R_n}{dr'^2} + r' \frac{dR_n}{dr'} - n^2 R_n(r') \right] \sin n\theta' = -2\pi r' \delta(r'-r)\,\delta(\theta'-\theta) \ ,$$

from which we can get the bracketed term as the coefficients of the sine Fourier series of the function in the right-hand side:

$$r'^2 \frac{d^2 R_n}{dr'^2} + r' \frac{dR_n}{dr'} - n^2 R_n(r') = \frac{2}{\pi} \int_0^\pi (-2\pi r')\, \delta(r'-r)\, \delta(\theta'-\theta) \sin n\theta' d\theta'$$
$$= (-4r \sin n\theta)\, \delta(r'-r) \ . \tag{4.5}$$

This is a differential equation showing that the expansion coefficients R_n are one-dimensional Green's functions in their own right, and we could even denote them by $R_n(r,\theta|r')$, that is, exhibiting r and θ, hidden variables in the right-hand side of (4.4). For $r' \neq r$, (4.5) is a homogeneous differential equation whose solution is that in (2.3), which, in this problem, cannot be written with the same arbitrary constants in both subdomains:

$$R_n(r') = \begin{cases} A_{1n} r'^n + B_{1n} r'^{-n} & (r' < r) \\ A_{2n} r'^n + B_{2n} r'^{-n} & (r' > r) \ . \end{cases}$$

We calculate the above arbitrary constants by imposing four conditions, which follow from those $G(r,\theta|r',\theta')$ must satisfy {cf. Ref. [3], sec. 12.2, and p. 522 in sec. 12.4; and Ref. [5], sec. 5.7.2}:

(i) Finiteness at the origin:

$$R_n(0) < \infty \ \Rightarrow \ B_{1n} = 0 \ \Rightarrow \ R_n(r') = \begin{cases} A_{1n} r'^n & (r' < r) \\ A_{2n} r'^n + B_{2n} r'^{-n} & (r' > r) \ . \end{cases}$$

(ii) The boundary condition at $r' = b$ which follows from (4.2b) using (4.4):

$$R_n'(b) = nA_{2n} b^{n-1} - nB_{2n} b^{-n-1} = 0 \ \Rightarrow \ R_n(r') = \begin{cases} A_{1n} r'^n & (r' < r) \\ A_{2n} (r'^n + b^{2n} r'^{-n}) & (r' > r) . \end{cases}$$

(iii) Continuity at $r' = r$: By letting $r^\pm$ to denote $r \pm \varepsilon$ with $\varepsilon \to 0^\pm$, we have

$$R_n(r^+) = R_n(r^-) \ \Rightarrow \ R_n(r') = \begin{cases} A_{2n} (r^n + b^{2n} r^{-n})\, r'^n / r^n & (r' \le r) \\ A_{2n} (r'^n + b^{2n} r'^{-n}) & (r' \ge r) \ . \end{cases}$$

(iv) The jump condition for the derivative at $r' = r$ [obtained by integrating (4.5) from $r' = r^-$ to $r' = r^+$]:

$$\frac{dR_n}{dr'}(r^+) - \frac{dR_n}{dr'}(r^-) = -\frac{4}{r} \sin n\theta$$
$$\Rightarrow \quad R_n(r') = \begin{cases} \dfrac{2}{n} \left[\left(\dfrac{r}{b}\right)^n + \left(\dfrac{b}{r}\right)^n \right] \left(\dfrac{r'}{b}\right)^n \sin n\theta & (r' < r) \\ \dfrac{2}{n} \left[\left(\dfrac{r'}{b}\right)^n + \left(\dfrac{b}{r'}\right)^n \right] \left(\dfrac{r}{b}\right)^n \sin n\theta & (r' > r) \ . \end{cases}$$

With the substitution of this result into (4.4), we finish the Green's function calculation:

$$G(r,\theta|r',\theta') = \sum_{n=1}^{\infty} \frac{2}{n}\left[\left(\frac{r_>}{b}\right)^n + \left(\frac{b}{r_>}\right)^n\right]\left(\frac{r_<}{b}\right)^n \sin n\theta \sin n\theta' \ , \tag{4.6}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r' (a definition that makes it possible to write $R_n(r')$, and therefore $G(r,\theta|r',\theta')$, with just one expression).

But using (2.7), we can eliminate the infinite summation in (4.6):

$$\begin{aligned} G(r,\theta|r',\theta') &= \sum_{n=1}^{\infty} \frac{2}{n}\left(\frac{r_>r_<}{b^2}\right)^n \sin n\theta \sin n\theta' + \sum_{n=1}^{\infty} \frac{2}{n}\left(\frac{r_<}{r_>}\right)^n \sin n\theta \sin n\theta' \\ &= \frac{1}{2}\ln\frac{(r_>r_<)^2 + b^4 - 2r_>r_<b^2\cos(\theta'+\theta)}{(r_>r_<)^2 + b^4 - 2r_>r_<b^2\cos(\theta'-\theta)} + \frac{1}{2}\ln\frac{r_<^2 + r_>^2 - 2r_<r_>\cos(\theta'+\theta)}{r_<^2 + r_>^2 - 2r_<r_>\cos(\theta'-\theta)} \ . \end{aligned}$$

Since $r_>r_< = rr'$ and $r_<^2 + r_>^2 = r^2 + r'^2$, we have

$$\begin{aligned} G(r,\theta|r',\theta') &= \frac{1}{2}\ln\frac{(rr')^2 + b^4 - 2rr'b^2\cos(\theta'+\theta)}{(rr')^2 + b^4 - 2rr'b^2\cos(\theta'-\theta)} \\ &\qquad + \frac{1}{2}\ln\frac{r^2 + r'^2 - 2rr'\cos(\theta'+\theta)}{r^2 + r'^2 - 2rr'\cos(\theta'-\theta)} \\ &= \frac{1}{2}\ln\frac{(rr'/b)^2 + b^2 - 2rr'\cos(\theta'+\theta)}{(rr'/b)^2 + b^2 - 2rr'\cos(\theta'-\theta)} \\ &\qquad + \frac{1}{2}\ln\frac{r^2 + r'^2 - 2rr'\cos(\theta'+\theta)}{r^2 + r'^2 - 2rr'\cos(\theta'-\theta)} \ . \end{aligned}$$

This expression is the same as that in (3.8b).

4.2 Reduction to a one-dimensional Green's function dependent on the angular variable

First of all, considering new variables ρ and ρ' that are respectively related to r and r' by the same law of transformation given in (2.9), and performing operations similar to those that led to the problem in (2.10), we can write the Green's function problem defined in (4.1) and (4.2) as follows:

$$\begin{cases} \dfrac{\partial^2 G}{\partial \rho'^2} + \dfrac{\partial^2 G}{\partial \theta'^2} = -2\pi\,\delta(\rho'-\rho)\,\delta(\theta'-\theta) \ \text{ with } \rho' \in (0,\infty) \ \text{ and } \ \theta' \in (0,\pi) \ , \\ G(\rho,\theta|\rho',\theta') = 0 \ \text{ if } \ \theta' = 0 \ \text{ or } \ \theta' = \pi \ \ \forall\, \rho' \in [0,\infty) \ , \\ \dfrac{\partial G}{\partial \rho'}(\rho,\theta|\rho',\theta') = 0 \ \text{ if } \ \rho' = 0 \ \ \forall\, \theta' \in (0,\pi) \ , \end{cases}$$

where we used the result $\delta(r'-r) = \delta(\rho'-\rho)|d\rho'/dr'| = \delta(\rho'-\rho)/r'$ (note the use of a well-known property of the delta function [1, eq. (1.180a)] in deducing this result).

Except for the product of delta functions on the right-hand side of the differential equation and the homogeneous condition at $\theta' = \pi$, this problem is similar to that in (2.10) and can also be solved by means of the cosine Fourier transform

$$\mathcal{F}_c\big\{G(\rho,\theta|\rho',\theta')\big\} = \sqrt{\frac{2}{\pi}}\int_0^\infty G(\rho,\theta|\rho',\theta')\cos k\rho'\, d\rho' \;\equiv\; \bar{G}(\rho,\theta|k,\theta') \;.$$

Using it to transform the partial differential equation and taking into account the boundary condition at $\rho' = 0$, we obtain

$$-k^2\bar{G}(\rho,\theta|k,\theta') - \sqrt{\frac{2}{\pi}}\;\underbrace{\left[\frac{\partial G}{\partial \rho}(\rho,\theta|\rho',\theta')\right]_{\rho'=0}}_{=0} + \frac{d^2\bar{G}}{d\theta'^2} = -2\pi\sqrt{\frac{2}{\pi}}\cos k\rho\;\delta(\theta'-\theta)\;. \quad (4.7)$$

This is a differential equation for the one-dimensional Green's function $\bar{G}(\rho,\theta|k,\theta')$. We solve it by a procedure analogous to the one adopted in solving (4.5). For $\theta' \neq \theta$, we have a homogeneous differential equation whose solution is

$$\bar{G}(\rho,\theta|k,\theta') = \begin{cases} A_{1k}\cosh k\theta' + B_{1k}\sinh k\theta' & (\theta' < \theta) \\ A_{2k}\cosh k\theta' + B_{2k}\sinh k\theta' & (\theta' > \theta) \end{cases} .$$

These two expressions for $\bar{G}(\rho,\theta|k,\theta')$ will lead to the desired $G(r,\theta|r',\theta')$ in two subdomains of Ω: a lower sector of the semidisc where $\theta' < \theta$ and a upper one where $\theta' > \theta$; this *sectorial* division of the semi-disc is depicted in Figure 7. Let us calculate the constants A_{1k}, etc., by imposing four conditions similar to those in Subsection 4.1:

(i) The boundary condition at $\theta' = 0$:

$$\bar{G}\Big|_{\theta'=0} = 0 \;\Rightarrow\; A_{1k} = 0 \;\Rightarrow\; \bar{G} = \begin{cases} B_{1k}\sinh k\theta' & (\theta' < \theta) \\ A_{2k}\cosh k\theta' + B_{2k}\sinh k\theta' & (\theta' > \theta) \end{cases} .$$

(ii) The boundary condition at $\theta' = \pi$:

$$\bar{G}\Big|_{\theta'=\pi} = 0 \;\Rightarrow\; A_{2k}\cosh k\pi + B_{2k}\sinh k\pi = 0 \;\Rightarrow\; B_{2k} = -A_{2k}\cosh k\pi/\sinh k\pi$$

$$\Rightarrow\; \bar{G}\Big|_{\theta'>\theta} = \underbrace{(A_{2k}/\sinh k\pi)}_{\equiv C_{2k}}(\sinh k\pi\cosh k\theta' - \cosh k\pi\sinh k\theta') = C_{2k}\sinh k(\pi-\theta')$$

$$\Rightarrow\; \bar{G}(\rho,\theta|k,\theta') = \begin{cases} B_{1k}\sinh k\theta' & (\theta' < \theta) \\ C_{2k}\sinh k(\pi-\theta') & (\theta' < \theta) \end{cases} .$$

(iii) Continuity at $\theta' = \theta$: By letting $\theta^\pm$ to denote $\theta \pm \varepsilon$ with $\varepsilon \to 0^\pm$, we have

$$\bar{G}\Big|_{\theta'\to\theta^+} = \bar{G}\Big|_{\theta'\to\theta^-} \;\Rightarrow\; \bar{G}(\rho,\theta|k,\theta') = \begin{cases} B_{1k}\sinh k\theta' & (\theta' \leq \theta) \\ B_{1k}\dfrac{\sinh k\theta\sinh k(\pi-\theta')}{\sinh k(\pi-\theta)} & (\theta' \geq \theta) \end{cases} .$$

(iv) The jump condition for the derivative at $\theta' = \theta$ [obtained by integrating (4.7) from $\theta' = \theta^-$ to $\theta' = \theta^+$]:

$$\frac{\partial \bar{G}}{\partial \theta'}\Big|_{\theta' \to \theta^+} - \frac{\partial \bar{G}}{\partial \theta'}\Big|_{\theta' \to \theta^-} = -2\pi\sqrt{\frac{2}{\pi}}\,\cos k\rho$$

$$\Rightarrow \quad \bar{G}(\rho,\theta|k,\theta') = \begin{cases} 2\pi\sqrt{\dfrac{2}{\pi}}\,\dfrac{\cos k\rho\,\sinh k\theta'\,\sinh k(\pi-\theta)}{k\sinh k\pi} & (\theta' \le \theta) \\[2ex] 2\pi\sqrt{\dfrac{2}{\pi}}\,\dfrac{\cos k\rho\,\sinh k\theta\,\sinh k(\pi-\theta')}{k\sinh k\pi} & (\theta' \ge \theta) \end{cases}$$

$$= 2\pi\sqrt{\frac{2}{\pi}}\,\frac{\cos k\rho\,\sinh k\theta_<\,\sinh k(\pi-\theta_>)}{k\sinh k\pi} \ ,$$

where $\theta_<$ ($\theta_>$) is the smaller (larger) of θ and θ'.

By taking the inverse cosine Fourier transform of this result, we finally determine Green's function:

$$\begin{aligned} G(\rho,\theta|\rho',\theta') &= \int_0^\infty \frac{4\cos k\rho\,\sinh k\theta_<\,\sinh k(\pi-\theta_>)}{k\sinh k\pi}\cos k\rho'\,dk \\ &= \frac{1}{2}\int_0^\infty \left[\cos k|\rho'+\rho| + \cos k|\rho'-\rho|\right]\frac{4\sinh k\theta_<\,\sinh k(\pi-\theta_>)}{k\sinh k\pi}\,dk \ . \end{aligned} \tag{4.8}$$

We did not find this integral listed (directly or indirectly) in any of the famous tables of integrals (it is more complicated than that indicated by I in (2.12) and listed in Ref. [6]), so we had to find a way to calculate it[2]; the method employed to accomplish it is presented in the Appendix, in equation (A.2).

We can write (4.8) as half the sum of two integrals of the type in (A.2), both with $B = \theta_<$, $C = \pi - \theta_>$, and $D = \pi$, but the first integral with $A = |\rho' + \rho|$ and the second one with $A = |\rho' - \rho|$. Therefore, since

$$\cosh\frac{\pi A}{D}\Big|_{A=|\rho'\pm\rho|} = \cosh(\rho' \pm \rho) = \cosh\!\left(\ln\frac{b}{r'} \pm \ln\frac{b}{r}\right) = \frac{1}{2rr'}\begin{cases} b^2 + (rr'/b^2)^2 & \text{if "+"} \\ r^2 + r'^2 & \text{if "−"} \end{cases}$$

$$\text{and} \quad \cos\frac{\pi(B \mp C)}{D} = \cos\left[\theta_< \mp (\pi - \theta_>)\right] = -\cos\left[\theta_< \pm \theta_>\right] = -\cos(\theta' \pm \theta) \ ,$$

using these results, we finally obtain

$$G = \frac{1}{2}\ln\frac{\dfrac{b^2 + (rr'/b^2)^2}{2rr'} - \cos(\theta'+\theta)}{\dfrac{b^2 + (rr'/b^2)^2}{2rr'} - \cos(\theta'-\theta)} + \frac{1}{2}\ln\frac{\dfrac{r^2 + r'^2}{2rr'} - \cos(\theta'+\theta)}{\dfrac{r^2 + r'^2}{2rr'} - \cos(\theta'-\theta)} \ ,$$

[2] At this point, concerned about the deadline and not knowing whether the method used here would work, and having found an article about a fairly comprehensive method for calculating definite integrals in Reference [8], we contacted its authors for help, who were very solicitous in calculating that integral, having obtained the same result (Robert Reynolds and Allan Stauffer, York University, Toronto – private communication, November 25th to December 2nd, 2021).

exactly the same Green's function expression given in (3.8b).

5 FINAL CONCLUSIONS

This works presents the calculation of the solution to the problem in (1.1) in terms of Green's function, which is here determined in three different ways with exactly the same form (in subsections 3.2, 4.1, and 4.2). It also presents the calculation of the solution to the particular case of (1.1) with $h(r,\theta) \equiv 0$ (Laplace's equation) by a method that combines separation of variables and the cosine Fourier transform (in Section 2), the solution being obtained with exactly the same form as the one calculated by the Green's function method.

We do not present the calculation of Poisson's equation solution by the method described in Section 2 because we could not write the part of its solution corresponding to the so-called source term [the function $h(r,\theta)$ in (1.1)] in the same form of (3.9), obtained using Green's function. That is the reason why Laplace's equation, in spite of being a particular case of Poisson's equation, is explicit in the title of the work.

We note the pleasant possibility of carrying out the two infinite summations in (2.6) and (4.6) as well as the two definite integrals in (2.12) and (4.8), without which we would not achieve the goal of writing the solution expressions resulting from different methods in exactly the same way. This pleasant possibility is exceptional, occurring only sporadically, but we can glimpse that it would happen in other problems with somewhat different geometry and boundary conditions; in fact, the deduction of (2.7) using (A.1) resembles that of the Poisson kernel [11, sec. VII.7].

Comparatively, the method of Green's function calculated by the method of images is superior because it leads more directly to a more improved form of solution, that is, closer to the "closed form". But this method has much less extensive application, because only in special situations can we determine the necessary images.

It is interesting to note the use of a Fourier transform in a *finite* domain problem, but a brief analysis shows that there is nothing new about this. Indeed; it should be noted that we performed the change of variable in (2.9), thus obtaining a problem in a *infinite* domain. Furthermore, the transformed problem can be solved by the alternative, but equivalent method of separation of variables. By this method, an eigenvalue problem of continuous spectrum in the radial variable is obtained. Therefore, to expand the solution in terms of eigenfunctions, it is necessary to perform not a summation, but an integration, which is the inverse of the Fourier transform used.

If we substitute $\theta = \pi$ directly into (2.13) with the intention of checking if that solution satisfies the boundary condition $u_2(r,\pi) = f(r)$, we would formally get the product $0 \cdot \int_0^b dr' f(r') \big[1/(r'-r)^2 + b^2/(rr'-b^2)^2\big]$, meaning that $\lim\limits_{\theta\to\pi} u_2(r,\theta)$ is an indeterminate form of the type $0 \times \infty$, because this is an improper integral that diverges [its integrand becomes infinite at $r' = r$]. The famous Poisson integral also exhibits this behavior on the boundary, and References [11, sec. VII, Theorem 10.1] and [9, Theorem 10.3.2], for example, prove that it satisfies the boundary condition. But proving that the obtained solution for the problem in (1.1) satisfies the boundary conditions (thus proving that it is continuous in $\Omega \cup \partial\Omega$) is beyond the scope of this work.

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A APPENDIX

In this Appendix, we develop formulas needed in the text. First, with the definition $z \equiv p\,e^{\mathrm{i}\varphi}$, from which $z^n = p^n e^{in\varphi} = p^n \cos n\varphi + \mathrm{i} p^n \sin n\varphi$, we evaluate the sum of the following infinite series:

$$\begin{aligned}
S_{\pm}(p,\varphi) &\equiv \sum_{n=1}^{\infty} \frac{(\pm 1)^{n+1}}{n}\, p^n \cos n\varphi \;=\; \sum_{n=1}^{\infty} \frac{(\pm 1)^{n+1}}{n}\, \mathrm{Re}\, z^n \\
&= \mathrm{Re} \sum_{n=1}^{\infty} (\pm 1)^{n+1} \frac{z^n}{n} \;=\; \mathrm{Re} \sum_{n=1}^{\infty} (\pm 1)^{n+1} \int_0^z \zeta^{\,n-1}\, d\zeta \\
&= \mathrm{Re} \int_0^z \Big[\sum_{n=0}^{\infty} (\pm 1)^n \zeta^{\,n} \Big] d\zeta \;=\; \mathrm{Re} \int_0^z \frac{d\zeta}{1 \mp \zeta} \;=\; \mathrm{Re}\big[\mp \ln(1 \mp z) \big] \\
&= \mp \mathrm{Re}\big[\ln|1 \mp z| + \mathrm{i} \arg(1 \mp z) \big] \;=\; \mp \ln|1 \mp z| \;=\; \mp \ln|1 \mp p\,e^{\mathrm{i}\varphi}| \\
&= \mp \ln|1 \mp p\cos\varphi \mp \mathrm{i} p \sin\varphi| \;=\; \mp \ln \sqrt{(1 \mp p\cos\varphi)^2 + (p\sin\varphi)^2} \\
&\quad = \mp \frac{1}{2} \ln(1 + p^2 \mp 2p\cos\varphi) \;, \quad \text{with } 0 \le p < 1 \;, \qquad \text{(A.1)}
\end{aligned}$$

where we used the known sum of the geometric series (justified by the fact that $|\zeta| \le |z| = p < 1$ along the straight path of integration from $\zeta = 0$ to $\zeta = z$) as well as the definition of the complex logarithm.

Below we perform the definite integral used to write the Green's function in (4.8) in closed form. The calculation is based on an usual application of the residue theorem: We express the integral (of an even function) as half its extension to the whole real axis, close its path with a semicircle C_R^+ of radius $R \to \infty$ in the upper half of the complex plane of $z = x + \mathrm{i}y$, assume that the integral over C_R^+ tends to zero according to Jordan's lemma, and evaluate the residues at the simple poles of the integrand inside the closed contour $C = [-R,R] \cup C_R^+$, that is, at the zeros $n\pi\mathrm{i}/D$ $(n = 1,2,3,\cdots)$ of $\sinh Dz$ ($z = 0$ is a removable singularity).

$$\begin{aligned}
&\int_0^{\infty} \cos Ax \frac{4\sinh Bx \sinh Cx}{x \sinh Dx}\, dx \;=\; \frac{1}{2}\, \mathrm{Re} \int_{-\infty}^{\infty} e^{\mathrm{i}Ax} \frac{4\sinh Bx \sinh Cx}{x \sinh Dx}\, dx \\
&= \frac{1}{2}\, \mathrm{Re} \Big(\oint_C - \overbrace{\int_{C_R^+}}^{\nearrow 0} \Big) e^{\mathrm{i}Az} \frac{4\sinh Bz \sinh Cz}{z \sinh Dz}\, dz \;=\; \frac{1}{2}\, \mathrm{Re} \Big[2\pi\mathrm{i} \sum_{n=1}^{\infty} \mathrm{Res}\Big(\frac{n\pi\mathrm{i}}{D} \Big) \Big] \\
&= \pi\, \mathrm{Re} \Big[\mathrm{i} \sum_{n=1}^{\infty} \lim_{z \to \frac{n\pi\mathrm{i}}{D}} \frac{z - \frac{n\pi\mathrm{i}}{D}}{\sinh Dz}\, e^{\mathrm{i}Az} \frac{4\sinh Bz \sinh Cz}{z} \Big] \\
&= \pi\, \mathrm{Re} \Big[\mathrm{i} \sum_{n=1}^{\infty} \frac{1}{D\cos n\pi}\, e^{-\frac{An\pi}{D}} \frac{4 \cdot \mathrm{i}\sin\frac{Bn\pi}{D} \cdot \mathrm{i}\sin\frac{Cn\pi}{D}}{\frac{n\pi\mathrm{i}}{D}} \Big]
\end{aligned}$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(e^{-\pi A/D}\right)^n \left[\cos\frac{n\pi(B-C)}{D} - \cos\frac{n\pi(B+C)}{D}\right]$$

$$= 2S_-\left(e^{-\pi A/D}, \frac{\pi(B-C)}{D}\right) - 2S_-\left(e^{-\pi A/D}, \frac{\pi(B+C)}{D}\right)$$

$$= \ln\frac{1+p^2+2p\cos\frac{\pi(B-C)}{D}}{1+p^2+2p\cos\frac{\pi(B+C)}{D}}\Bigg|_{p=e^{-\frac{\pi A}{D}}} = \ln\frac{\frac{1}{2}\left(\frac{1}{p}+p\right)+\cos\frac{\pi(B-C)}{D}}{\frac{1}{2}\left(\frac{1}{p}+p\right)+\cos\frac{\pi(B+C)}{D}}\Bigg|_{p=e^{-\frac{\pi A}{D}}}$$

$$= \ln\frac{\cosh\frac{\pi A}{D}+\cos\frac{\pi(B-C)}{D}}{\cosh\frac{\pi A}{D}+\cos\frac{\pi(B+C)}{D}}\ , \quad \text{with } A\geq 0,\ D>0,\ B+C\leq D\ , \tag{A.2}$$

where the sums of the infinite series were obtained by using (A.1). The restrictions upon the parameters A, B, C, and D guarantee the convergence of the integral and the use of Jordan's lemma {cf. Ref. [1, eq.(7.43)]}.