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A Note on C^2 Ill-posedness Results for the Zakharov System in Arbitrary Dimension

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ABSTRACT. This work is concerned with the Cauchy problem for a Zakharov system with initial data in Sobolev spaces $H^k(\mathbb{R}^d)\times H^l(\mathbb{R}^d)\times H^{l-1}(\mathbb{R}^d)$. We recall the well-posedness and ill-posedness results known to date and establish new ill-posedness results. We prove C^2 ill-posedness for some new indices $(k,l)\in\mathbb{R}^2$. Moreover, our results are valid in arbitrary dimension. We believe that our detailed proofs are built on a methodical approach and can be adapted to obtain similar results for other systems and equations.

Keywords: Zakharov System, C^2 Ill-posedness

1 INTRODUCTION

This work is concerned with the Cauchy problem for the following Zakharov system

$$\begin{cases} i\partial_t u + \Delta u = nu\,, & u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}\,, \\ \partial_t^2 n - \Delta n = \Delta|u|^2, & n:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}, \\ (u,n,\partial_t n)|_{t=0}\in H^{k,l}, & \end{cases} \tag{Z}$$

where $H^{k,l}$ is a short notation for the Sobolev space $H^k(\mathbb{R}^d;\mathbb{C})\times H^l(\mathbb{R}^d;\mathbb{R})\times H^{l-1}(\mathbb{R}^d;\mathbb{R})$, $(k,l)\in\mathbb{R}^2$ and Δ is the laplacian operator for the spatial variable.

V. E. Zakharov introduced the system (Z) in [19] to describe the long wave Langmuir turbulence in a plasma. The function u represents the slowly varying envelope of the rapidly oscillating electric field and the function n denotes the deviation of the ion density from its mean value.

In this note we prove that, for any dimension d, the system (Z) is C^2 ill-posed in $H^{k,l}$, for the indices (k,l) displayed in Figure 1 and Figure 2 (see Theorem 1.2 and Theorem 1.3 for the precise statements). The first C^2 *ill-posedness* result was proved by Tzvetkov in [18] for the KdV

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equation, improving the previous C^3 *ill-posedness* result of Bourgain found in [6]. We essentially follow the same ideas of [18], but our proofs are structured as in [9]. Two slightly different senses of C^2 ill-posedness are considered in our results (see also Remark 1).

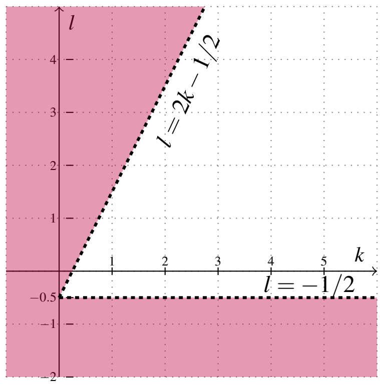


Figure 1: S^t is not C^2. Theorem 1.2.

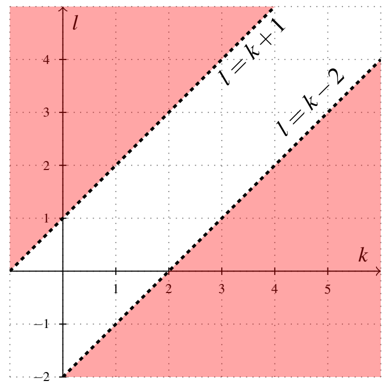


Figure 2: S is not C^2. Theorem 1.3.

Ginibre, Tsutsumi and Velo introduced in [11] a heuristic critical regularity for the system (Z), which is given by $(k,l)=(d/2-3/2, d/2-2)$. In particular, our result in Theorem 1.2 with $d=3$ (physical dimension) shows that the critical regularity $(0,-1/2)$ is the endpoint for achieving well-posedness by fixed point procedure. We point out that local well-posedness at critical regularity is an open problem for $d \geq 3$.

The system (Z) has been studied in several works. Bourgain and Colliander proved in [7] local well-posedness in the energy norm for $d=2,3$. They construct local solutions applying the contraction principle in $X^{s,b}$ spaces introduced in [5]. Local well-posedness in arbitrary dimension under weaker regularity assumptions was obtained in [11] by Ginibre, Tsutsumi and Velo. We recall the last result in the next theorem (see Figure 3).

Theorem 1.1. *(Ginibre, Tsutsumi and Velo [11]) Let $d \geq 1$. The system* (Z) *is locally well-posed, provided*

$$\begin{array}{lll} -1/2 < k-l \leq 1, & 2k \geq l+1/2 \geq 0, & \textit{for } d=1 \\ l \leq k \leq l+1, & & \textit{for all } d \geq 2 \\ l \geq 0, & 2k-(l+1) \geq 0, & \textit{for } d=2,3 \\ l > d/2-2, & 2k-(l+1) > d/2-2, & \textit{for all } d \geq 4. \end{array} \tag{1.1}$$

Now, we list the best results to date (as far as we know) for the system (Z).

For $d=1$, Theorem 1.1 is the best result for l.w.p. Concerning ill-posedness: Biagioni and Linares proved in [4] non-existence of uniformly continuous *solution mapping*, for $k<0$ and $l \leq -3/2$; Holmer proved in [12] norm inflation for $0<k<1$ and $l > 2k-1/2$ and for $k \leq 0$ and $l > -1/2$; Also in [12], non-existence of uniformly continuous *solution mapping* is proved

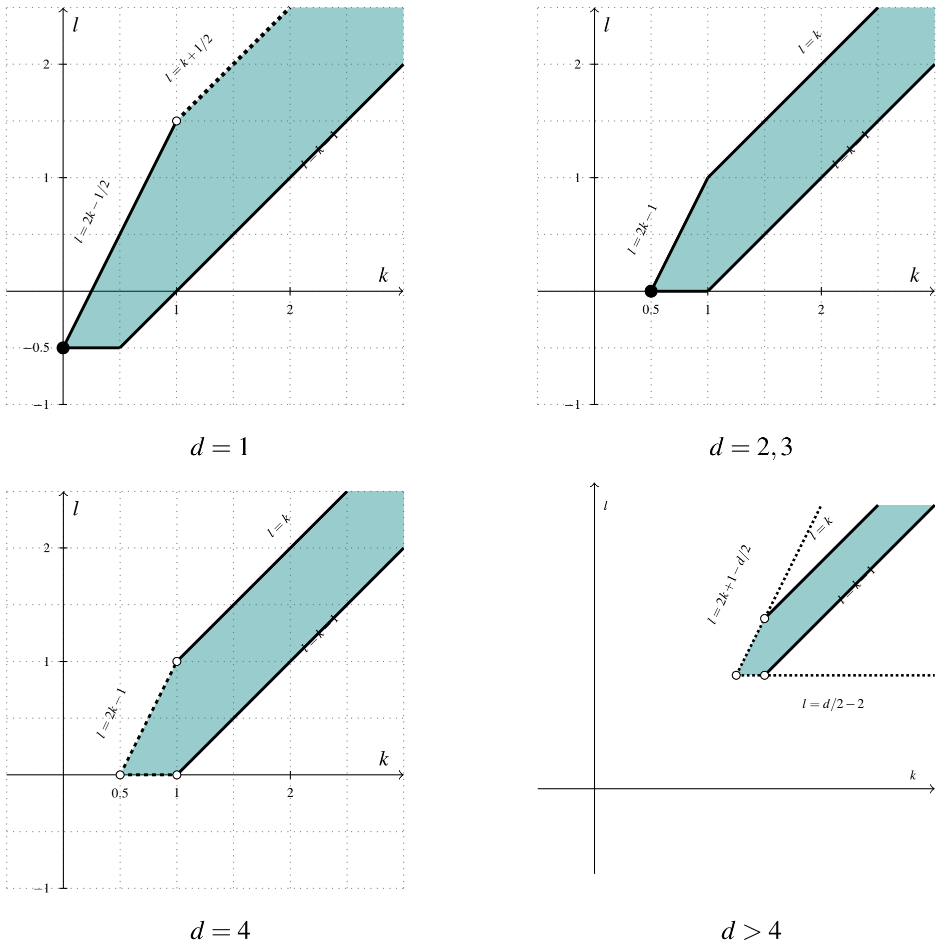


Figure 3: Regions corresponding to (1.1) for each case of dimension d.

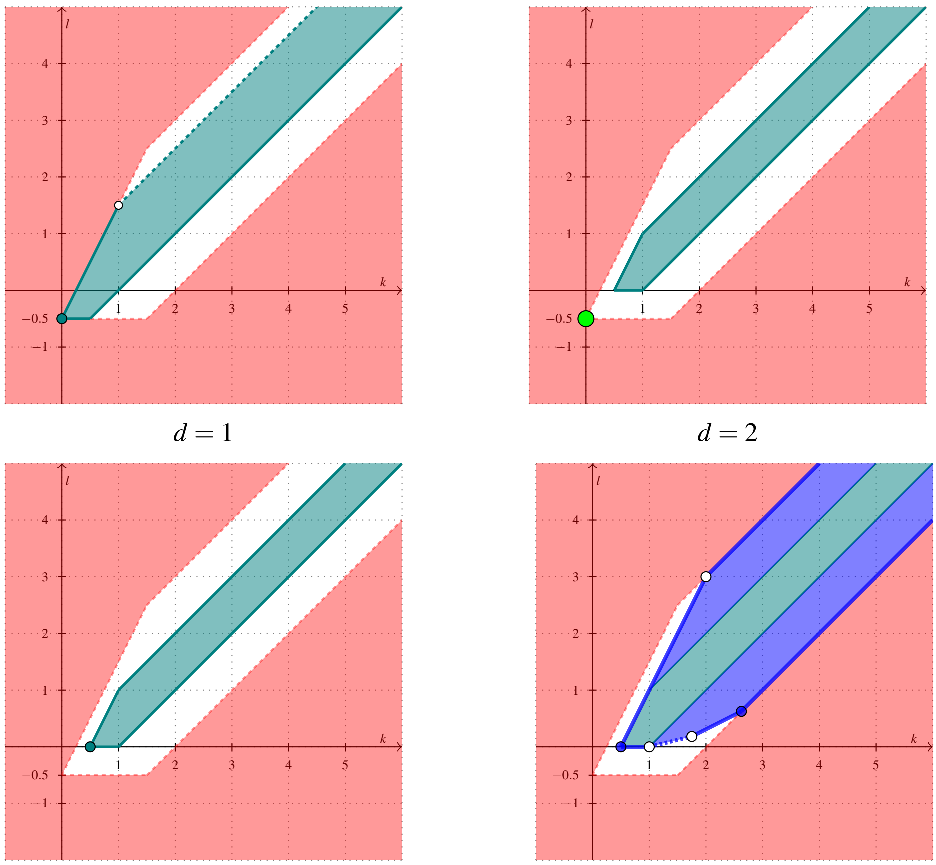
for $k = 0$ and $l < -3/2$; Theorem 1.2 (see Remark 1) and Theorem 1.3 are the best results for the remaining region.

For $d = 2$, Bejenaru, Herr, Holmer and Tataru in [2] proved l.w.p. for $(k,l) = (0,-1/2)$ and Theorem 1.1 is the best result for the remaining indices k and l. Concerning ill-posedness, Theorem 1.2 (see Remark 1) and Theorem 1.3 are the best results.

For $d = 3$, Theorem 1.1 is the best result for l.w.p. Concerning ill-posedness: Theorem 1.2 and Theorem 1.3 are the best results.

For $d = 4$, Bejenaru, Guo, Herr and Nakanishi in [1] proved l.w.p. for $l \geq 0$, $k < 4l + 1$, $\max\{(l+1)/2, l-1\} \leq k \leq \min\{l+2, 2l+11/8\}$ and $(k,l) \neq (2,3)$. Theorem 1.1 is the best result for the remaining indices k and l. Concerning ill-posedness: Non-existence of solution is also proved in [1]. Theorem 1.2 (see Remark 1) and Theorem 1.3 are the best results for the remaining indices k and l.

For $d > 4$, Theorem 1.1 is the best result for l.w.p. Concerning ill-posedness: Theorem 1.2 and Theorem 1.3 are the best results. The next figure illustrates all these results.



$d = 1$ $d = 2$

$d = 3$ $d = 4$

Figure 4: l.w.p. Thm 1.1 l.w.p. [2] l.w.p. [1] ill-p. (at least C^2).

For $d \geq 4$, Kato and Tsugawa in [13] proved the global well-posedness of the Zakharov system for small data in the mixed inhomogeneous and homogeneous space $H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d) \times \dot{H}^{l-1}(\mathbb{R}^d)$ at critical regularity $(k,l) = (d/2 - 3/2, d/2 - 2)$. Global well-posedness for the Zakharov system is also studied in [16], [17], [8], [10], [15] and [1].

Now we start to state our results. First, we outline some definitions. Assume that the system (Z) is locally well-posed in the time interval $[0,T]$. Then the *solution mapping* associated to the system (Z) is the following map

$$\begin{array}{rcccl} S & : & B_r & \longrightarrow & \mathscr{C}(\,[0,T]\,;\,H^{k,l}\,) \\ & & (\varphi,\psi,\phi) & \mapsto & (u_{(\varphi,\psi,\phi)}\,,\,n_{(\varphi,\psi,\phi)}\,,\,\partial_t n_{(\varphi,\psi,\phi)}\,), \end{array} \tag{1.2}$$

where $\mathscr{C}([0,T]\;;\;H^{k,l})$ is a short notation for $C([0,T];H^k(\mathbb{R}^d))\times C([0,T];H^l(\mathbb{R}^d))\times C([0,T];H^{l-1}(\mathbb{R}^d))$,
$B_r=\{(\varphi,\psi,\phi)\in H^{k,l}\,:\,\|(\varphi,\psi,\phi)\|_{H^{k,l}}<r\}$ and $u_{(\varphi,\psi,\phi)}$ and $n_{(\varphi,\psi,\phi)}$ are local solutions[1] for system (Z) with initial data $(u,v,\partial_t n)|_{t=0}=(\varphi,\psi,\phi)$.

Since Theorem 1.1 was obtained by means of contraction method, one can conclude the following: If (k,l) satisfies conditions (1.1) then for every fixed $r>0$ there is a $T=T(r,k,l)>0$ such that the *solution mapping* (1.2) is analitic (see Theorem. 3 in [3]). So, if the system (Z) is locally well-posed in $H^{k,l}$ and the *solution mapping* (1.2) fails to be m-times differentiable, then the usual contraction method can not be applied to prove the local well-posedness. In this case, we have a sense of ill-posedness and we say that the system (Z) is ill-posed by the method or simply the system (Z) is C^m *ill-posed*[2]

in $H^{k,l}$.

Now fix $t\in[0,T]$. Hereafter we call *flow mapping* associated to the system (Z) the following map

$$\begin{array}{rcccl} S^t & : & B_r & \longrightarrow & H^k(\mathbb{R}^d)\times H^l(\mathbb{R}^d)\times H^{l-1}(\mathbb{R}^d) \\ & & (\varphi,\psi,\phi) & \mapsto & \Big(u_{(\varphi,\psi,\phi)}(t)\,,\,n_{(\varphi,\psi,\phi)}(t)\,,\,\partial_t n_{(\varphi,\psi,\phi)}(t)\Big). \end{array} \tag{1.3}$$

We are now ready to enunciate our results. Our first theorem shows that, in any dimension, the regularity $(k,l)=(0,-1/2)$ is the endpoint for achieving well-posedness by contraction method (see Figure 1).

Theorem 1.2. *Let $d\in\mathbb{N}$. Assume that the system* (Z) *is locally well-posed in the time interval* $[0,T]$. *For any fixed $t\in(0,T]$, the flow mapping* (1.3) *fails to be C^2 at the origin in $H^{k,l}$, provided $l<-1/2$ or $l>2k-1/2$.* According to [11] (see p. 387), the optimal relation between k and l is $l-k+1/2=0$. Our next theorem shows that when $|l-k+1/2|>3/2$ (i.e., $l<k-2$ or $l>k+1$) the system (Z) is C^2 ill-posed (see Figure 2).

Theorem 1.3. *Let $d\in\mathbb{N}$. Assume that the system* (Z) *is locally well-posed in the time interval* $[0,T]$. *The solution mapping* (1.2) *fails to be C^2 at the origin in $H^{k,l}$, provided $l<k-2$ or $l>k+1$.*

[1] Precisely, $u_{(\varphi,\psi,\phi)}$, $n_{(\varphi,\psi,\phi)}$, $\partial_t n_{(\varphi,\psi,\phi)}$ satisfy the integral equations (3.1), (3.2), (3.3) associated to the system (Z), for all $t\in[0,T]$.
[2] Actually, C^m ill-posedness means that the *solution mapping* is not m-times Fréchet differentiable.

Remark 1. *The sense of ill-posedness stated in Theorem 1.2 is slightly stronger than the sense stated in Theorem 1.3. Indeed, if the flow mapping* (1.3) *is not C^2, neither is, a fortiori, the solution mapping* (1.2). *Thus, Theorem 1.2 slightly improves the ill-posedness results in [12] and [2], for $d = 1$ and $d = 2$, respectively, both establishing that the solution mapping* (1.2) *is not C^2 for $l < -1/2$ or $l > 2k - 1/2$.*

Remark 2. *Theorem 1.3 establishes C^2 ill-posedness for new indices (k,l) (see Figure 2). For such indices, the difference of regularity between the initial data is large (i.e., $l \gg k$ or $k \gg l$). Such result seems natural, due to coupling of the system via nonlinearities. Indeed, for instance, high regularity for $u(t)$ is not expect when $n(t)$ has low regularity, in view of* (3.1). *By the way, the C^2 ill-posedness for $l < k-2$ is obtained by dealing with* (3.1).

Remark 3. *In the periodic setting, Kishimoto proved in [14] the C^2 ill-posedness*[3] *of the Zakharov system in $H^k(\mathbb{T}^d)\times H^l(\mathbb{T}^d)\times H^{l-1}(\mathbb{T}^d)$ for $d \geq 2$, provided $l < \max\{0\,,k-2\}$ or $l > \min\{2k-1\,,k+1\}$. These indices (k,l) are exactly the same of Theorems 1.2 and 1.3, excepting for admitting $-1/2 \leq l < 0$. We point out that in [2] was proved, by means of contraction method, that the system* (Z) *is locally well-posed for $d = 2$, $k = 0$ and $l = -1/2$.*

This paper is organized as follows. In Section 2, we introduce some notations to be used throughout the whole text. In Section 3, is presented a preliminary analysis which provides a methodical approach to our proofs, exposing the main ideas. In Section 4, we prove Theorem 1.2 and in Section 5, we prove Theorem 1.3.

2 NOTATIONS

- $(*.*)_R$ (or $(*.*)_L$) denotes the right(or left)-hand side of an equality or inequality numbered by $(*.*)$.
- $\|(\varphi,\psi,\phi)\|^2_{H^{k,l}} = \|\varphi\|^2_{H^k} + \|\psi\|^2_{H^l} + \|\phi\|^2_{H^{l-1}}$, where $H^{k,l} = H^k(\mathbb{R}^d;\mathbb{C}) \times H^l(\mathbb{R}^d;\mathbb{R}) \times H^{l-1}(\mathbb{R}^d;\mathbb{R})$.
- $\langle\xi\rangle = \sqrt{1+|\xi|^2}\,,\ \xi \in \mathbb{R}^d$.
- χ_Ω denotes the characteristic function of $\Omega \subset \mathbb{R}^d$.
- $|\Omega|$ denotes de Lebesgue measure of the set Ω, i.e., $|\Omega| = \int \chi_\Omega(\xi)d\xi$.
- $\mathscr{S}(\mathbb{R}^d)$ denotes the Schwartz space and $\mathscr{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions.
- $\widehat{f}$ and $\check{f}$ denote, respectively, the Fourier transform and the inverse Fourier transform of $f \in \mathscr{S}'(\mathbb{R}^d)$.

[3] C^2 ill-posedness in the slightly weaker sense (see Remark 1). However, for $d = 2$ and particular (k,l) is proved in [14] ill-posedness in much stronger senses, namely norm inflation and non-existence of continuous *solution mapping*.

3 PRELIMINARY ANALYSIS

The integral equations associated to the system (Z) with initial data $(u,v,\partial_t n)|_{t=0} = (\varphi,\psi,\phi)$ are

$$u(t) = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}u(s)n(s)ds, \tag{3.1}$$

$$n(t) = W(t)(\psi,\phi) + \int_0^t W_1(t-s)\Delta|u|^2(s)ds, \tag{3.2}$$

$$\partial_t n(t) = W(t)(\phi,\Delta\psi) + \int_0^t W_0(t-s)\Delta|u|^2(s)ds, \tag{3.3}$$

where $\{e^{it\Delta}\}_{t\in\mathbb{R}}$ is the unitary group in $H^s(\mathbb{R}^d)$ associated to the linear Schrödinger equation, given by $e^{it\Delta}\varphi := \{e^{-it|\cdot|^2}\widehat{\varphi}(\cdot)\}^{\vee}$ and $\{W(t)\}_{t\in\mathbb{R}}$ is the linear wave propagator $W(t)(\psi,\phi) := W_0(t)\psi + W_1(t)\phi$, where W_0 and W_1 are given by $W_0(t)\psi = \cos\left(t\sqrt{-\Delta}\right)\psi := \{\cos(t|\cdot|)\widehat{\psi}(\cdot)\}^{\vee}$ and $W_1(t)\phi = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\phi := \left\{\frac{\sin(t|\cdot|)}{|\cdot|}\widehat{\phi}(\cdot)\right\}^{\vee}$.

Assume that the system (Z) is locally well-posed in $H^{k,l}$, in the time interval $[0,T]$. Suppose also that there exists $t\in[0,T]$ such that the *flow mapping* (1.3) is two times Fréchet differentiable at the origin in $H^{k,l}$. Then, the second Fréchet derivative of S^t at origin belongs to $\mathscr{B}$, the normed space of bounded bilinear applications from $H^{k,l}\times H^{k,l}$ to $H^{k,l}$. In particular, we have the following estimate for the second Gâteaux derivative of S^t at origin

$$\left\| \frac{\partial S^t_{(0,0,0)}}{\partial\Phi_0\partial\Phi_1} \right\|_{H^{k,l}} = \left\| D^2 S^t_{(0,0,0)}(\Phi_0,\Phi_1) \right\|_{H^{k,l}} \leq \left\| D^2 S^t_{(0,0,0)} \right\|_{\mathscr{B}} \|\Phi_0\|_{H^{k,l}} \|\Phi_1\|_{H^{k,l}} \tag{3.4}$$

for all $\Phi_0,\Phi_1\in H^{k,l}$. Similarly, assuming *solution mapping* (1.2) two times Fréchet differentiable at the origin, we have $D^2 S_{(0,0,0)}$ belonging to $\mathscr{B}_{\mathscr{C}}$, the normed space of bounded bilinear applications from $H^{k,l}\times H^{k,l}$ to $\mathscr{C}([0,T];H^{k,l})$. Then

$$\sup_{t\in[0,T]} \left\| \frac{\partial S^t_{(0,0,0)}}{\partial\Phi_0\partial\Phi_1} \right\|_{H^{k,l}} \leq \left\| D^2 S_{(0,0,0)} \right\|_{\mathscr{B}_{\mathscr{C}}} \|\Phi_0\|_{H^{k,l}} \|\Phi_1\|_{H^{k,l}}, \qquad \forall\Phi_0,\Phi_1\in H^{k,l}. \tag{3.5}$$

Thus, we can prove Theorem 1.2 by showing that estimate (3.4) is false for (k,l) in the region of Figure 1. In the case of Theorem 1.3, the indices (k,l) in the region of Figure 2 impose additional technical difficulties to get good lower bounds for $(3.4)_L$. To overcome such difficulties, we made use of a sequence $t_N\to 0$, in consequence, we merely prove that estimate (3.5) is false, obtaining an ill-posedness result in a slightly weaker sense.

Since $S^t_{(0,0,0)} = (0,0,0)$, for each direction $\Phi = (\varphi,\psi,\phi)\in\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)$, the first Gâteaux derivatives of $(3.1)_R$, $(3.2)_R$ and $(3.3)_R$ at the origin are $e^{it\Delta}\varphi$, $W(t)(\psi,\phi)$ and $W(t)(\phi,\Delta\psi)$, respectively. Further, from (3.4), we deduce the following estimates for the second Gâteaux derivatives of $u(t)$, $n(t)$ and $\partial_t n(t)$ in the directions $(\Phi_0,\Phi_1) = ((\varphi_0,\psi_0,\phi_0),(\varphi_1,\psi_1,\phi_1)) \in (\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d))^2$

$$\begin{aligned}\left\|\frac{\partial^2 u_{(0,0,0)}}{\partial\Phi_0\partial\Phi_1}(t)\right\|_{H^k} &= \left\|\int_0^t e^{i(t-s)\Delta}\{e^{is\Delta}\varphi_0 W(s)(\psi_1,\phi_1)+e^{is\Delta}\varphi_1 W(s)(\psi_0,\phi_0)\}ds\right\|_{H^k}\\ &\lesssim \|\Phi_0\|_{H^{k,l}}\|\Phi_1\|_{H^{k,l}},\end{aligned} \tag{3.6}$$

$$\begin{aligned}\left\|\frac{\partial^2 n_{(0,0,0)}}{\partial\Phi_0\partial\Phi_1}(t)\right\|_{H^l} &= \left\|\int_0^t W_1(t-s)\Delta\{e^{is\Delta}\varphi_0\overline{e^{is\Delta}\varphi_1}+\overline{e^{is\Delta}\varphi_0}e^{is\Delta}\varphi_1\}ds\right\|_{H^l}\\ &\lesssim \|\Phi_0\|_{H^{k,l}}\|\Phi_1\|_{H^{k,l}},\end{aligned} \tag{3.7}$$

$$\begin{aligned}\left\|\frac{\partial^2 \partial_t n_{(0,0,0)}}{\partial\Phi_0\partial\Phi_1}(t)\right\|_{H^{l-1}} &= \left\|\int_0^t W_0(t-s)\Delta\{e^{is\Delta}\varphi_0\overline{e^{is\Delta}\varphi_1}+\overline{e^{is\Delta}\varphi_0}e^{is\Delta}\varphi_1\}ds\right\|_{H^{l-1}}\\ &\lesssim \|\Phi_0\|_{H^{k,l}}\|\Phi_1\|_{H^{k,l}}.\end{aligned} \tag{3.8}$$

Hence, the proof of Theorem 1.2 boils down to getting sequences of directions Φ showing that one of these last three estimates fails for the fixed $t\in[0,T]$. For Theorem 1.3, such sequences just need to show that one of (3.6)-(3.8) can not hold uniformly for $t\in[0,T]$.

We deal with (3.6) by choosing directions $\Phi_0=\Phi_1=(\varphi,\psi,0)$ with $\varphi,\psi\in S(\mathbb{R}^d)$. Since in $\mathscr{S}(\mathbb{R}^d)$ the Fourier transform convert products in convolutions, from (3.6) we conclude the following estimate

$$\left\|\langle\xi\rangle^k\int_0^t e^{-i(t-s)|\xi|^2}\int_{\mathbb{R}^d} e^{-is|\xi_1|^2}\widehat{\varphi}(\xi_1)\cos(s|\xi-\xi_1|)\widehat{\psi}(\xi-\xi_1)d\xi_1 ds\right\|_{L^2_\xi} \lesssim \|\varphi\|^2_{H^k}+\|\psi\|^2_{H^l}, \tag{3.9}$$

for all $\varphi,\psi\in\mathscr{S}(\mathbb{R}^d)$. Hereafter we will denote, as usual, $\xi_2 := \xi-\xi_1$, then

$$\xi_1+\xi_2 = \xi\,. \tag{3.10}$$

For bounded subsets $A,B\subset\mathbb{R}^d$, by taking $\varphi,\psi\in\mathscr{S}(\mathbb{R}^d)$ such that[4] $\langle\cdot\rangle^k\,\widehat{\varphi}\sim\chi_A$ and $\langle\cdot\rangle^l\,\widehat{\psi}\sim\chi_B$, we conclude from (3.9) that

$$\left\|\int_0^t\int_{\mathbb{R}^d}\frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l}\cos(s|\xi|^2-s|\xi_1|^2)\cos(s|\xi_2|)\chi_A(\xi_1)\chi_B(\xi_2)d\xi_1 ds\right\|_{L^2_\xi}\lesssim |A|+|B|. \tag{3.11}$$

We can rewrite $(3.11)_L$ as

$$\left\|\int_0^t\int_{\mathbb{R}^d}\frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l}\,\tfrac{1}{2}\left[\cos(\sigma_+ s)+\cos(\sigma_- s)\right]\chi_A(\xi_1)\chi_B(\xi_2)d\xi_1 ds\right\|_{L^2_\xi}, \tag{3.12}$$

[4]Precisely, $\chi_A\le\langle\cdot\rangle^k\,\widehat{\varphi}$ with $\|\varphi\|_{H^k}\le 2\|\chi_A\|_{L^2}$ and $\chi_B\le\langle\cdot\rangle^l\,\widehat{\psi}$ with $\|\psi\|_{H^l}\le 2\|\chi_B\|_{L^2}$.

where σ_+ and σ_+ are what we call *the algebraic relations* associated to (3.6), given by

$$\sigma_\pm \;:=\; |\xi|^2 - |\xi_1|^2 \pm |\xi_2|. \tag{3.13}$$

Finally, we have to choose sequences of sets $\{A_N\}_{N\in\mathbb{N}}$ and $\{B_N\}_{N\in\mathbb{N}}$ such that, for $\xi_1 \in A_N$ and $\xi_2 \in B_N$, yields increasing $\frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l}$, small σ_+ and large σ_-, when $N \to +\infty$. It allows us to get good lower bounds for (3.12), since

$$\cos(\theta) > 1/2, \quad \forall\theta \in (-1,1) \qquad \text{and} \qquad \int_0^t \cos(ks)ds = \frac{\sin(kt)}{k}, \quad \forall k \neq 0. \tag{3.14}$$

Moreover, we will need a lower bound for $\|\chi_{A_N} * \chi_{B_N}\|_{L^2}$. For this purpose, the next elementary result is very useful.

Lemma 3.1. *([9]) Let $A,B,R \subset \mathbb{R}^d$. If $R - B = \{x - y \;:\; x \in R \text{ and } y \in B\} \subset A$ then*

$$|R|^{\frac{1}{2}}|B| \leq \|\chi_A * \chi_B\|_{L^2(\mathbb{R}^d)}.$$

Remark 1. *For the case $l < -1/2$ in Theorem 1.2, by a good choice of A_N and B_N, it is possible to obtain a "high + high = high" interaction in* (3.10) *providing "high" $\frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l}$, "low" σ_+ and "high" σ_-, which yield good lower bounds for* (3.12). *But for the case $k - l > 2$ in Theorem 1.3, to obtain "high" $\frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l}$, the interaction must be of type "low + high = high", implying "high" σ_+ and "high" σ_-, which do not provide lower bound for* (3.12). *Then we choose a sequence $t_N \to 0$, allowing us to obtain lower bounds directly from* $(3.11)_L$.

4 PROOF OF THEOREM ??

Assume that, for a fixed $t \in (0,T]$, the *flow mapping* (1.3) is C^2 at the origin. Then, from (3.11), (3.12) and (3.13), we get the following estimate for bounded subsets $A,B \subset \mathbb{R}^d$

$$\|I^+_{A,B}(\xi)\|_{L^2_\xi} \;-\; \|I^-_{A,B}(\xi)\|_{L^2_\xi} \;\lesssim\; |A| + |B|, \tag{4.1}$$

where

$$I^\pm_{A,B}(\xi) \;:=\; \int_0^t \int_{\mathbb{R}^d} \frac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l} \cos(\sigma_\pm s)\chi_A(\xi_1)\chi_B(\xi_2)d\xi_1 ds. \tag{4.2}$$

Note that, for $\xi_1 = (\xi_1^1, \cdots, \xi_1^d) \in \mathbb{R}^d$ and $\xi_2 = (\xi_2^1, \cdots, \xi_2^d) \in \mathbb{R}^d$, we can rewrite (3.13) as

$$\sigma_\pm \;=\; \sum_{j=1}^d \Big(|\xi_1^j + \xi_2^j|^2 - |\xi_1^j|^2\Big) \;\pm\; |\xi_2| \;=\; \xi_2^1(2\xi_1^1 + \xi_2^1 \pm 1) \;\pm\; (|\xi_2| - \xi_2^1) \;+\; \sum_{j=2}^d \xi_2^j(2\xi_1^j + \xi_2^j). \tag{4.3}$$

In order to obtain a lower bound for $\|I^+_{A,B}\|_{L^2}$ and an upper bound $\|I^-_{A,B}\|_{L^2}$, we choose the sets $A,B \subset \mathbb{R}^d$ taking (4.3) into account. So, for $N \in \mathbb{N}$ and $0 < \delta < \min\{\frac{1}{7t}, 1\}$, we define[5]

[5]Evidently, if $d = 1$ then A and B are just intervals, the last sum in (4.3) does not exist and $(4.6)_R$ should be ignored.

$$A = A_N \ := \ \Big[-N\,,\,-N+\tfrac{\delta}{N}\Big] \times \Big[0\,,\,\tfrac{\delta}{d-1}\Big]^{d-1}$$

and

$$B = B_N \ := \ \Big[2N-1\,,\,2N-1+\tfrac{\delta}{2N}\Big] \times \Big[0\,,\,\tfrac{\delta}{2(d-1)}\Big]^{d-1}.$$

Then, for $(\xi_1,\xi_2) \in A_N \times B_N$, we have

$$\langle \xi_1 \rangle \ \sim \ \langle \xi_2 \rangle \ \sim \ \langle \xi_1+\xi_2 \rangle \ \sim \ N \tag{4.4}$$

and since $\delta < 1$ we also have $\xi_2^1 \in [N\,,\,2N]$ and $(2\xi_1^1+\xi_2^1) \in [-1\,,\,-1+\frac{5\delta}{2N}]$. Thus,

$$\xi_2^1(2\xi_1^1+\xi_2^1+1) \in \ [0\,,\,5\delta]\,, \qquad \xi_2^1(2\xi_1^1+\xi_2^1-1) \in \ [-4N\,,\,-N]\,, \tag{4.5}$$

$$(|\xi_2|-\xi_2^1) \in \Big[0\,,\,\tfrac{\delta}{2}\Big] \qquad \text{and} \qquad \sum_{j=2}^{d} \xi_2^j(2\xi_1^j+\xi_2^j) \in \Big[0\,,\,\tfrac{5\delta^2}{4(d-1)}\Big]. \tag{4.6}$$

Therefore, combining (4.3), $(4.5)_L$ and (4.6) we obtain

$$\sigma_+ \in [0\,,\,7\delta) \tag{4.7}$$

and combining (4.3), $(4.5)_R$ and (4.6) we obtain

$$\sigma_- \in \big(-5N\,,\,-\tfrac{1}{2}N\big)\,. \tag{4.8}$$

Since $\delta < \frac{1}{7t}$, from (4.7) and (3.14), we have $\cos(\sigma_+ s) > 1/2$. Moreover, from (4.4), yields $\frac{<\xi>^k}{<\xi_1>^k<\xi_2>^l} \sim N^l$. Hence, we conclude from (4.2) that

$$I^+_{A,B}(\xi) \ \geq \ \tfrac{1}{2}\int_0^t \int_{\mathbb{R}^d} \tfrac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l}\chi_A(\xi_1)\chi_B(\xi_2)d\xi_1 ds \ \gtrsim \ tN^{-l}\chi_A * \chi_B(\xi). \tag{4.9}$$

Now, Lemma 3.1 allows us to get a lower bound for $I^+_{A,B}(\xi)$. For this purpose, consider the set

$$R = R_N \ := \ \Big[N-1+\tfrac{\delta}{2N}\,,\,N-1+\tfrac{\delta}{N}\Big] \times \Big[\tfrac{\delta}{2(d-1)}\,,\,\tfrac{\delta}{d-1}\Big]^{d-1}.$$

Then we have $R-B \subset A$. Also, computing the Lebesgue measure of these cartesian products of intervals, we have

$$|R| \sim |A| \sim |B| \sim N^{-1}. \tag{4.10}$$

Using (4.9), Lemma 3.1 and (4.10) we obtain that

$$\|I^+_{A,B}\|_{L^2} \ \gtrsim \ tN^{-l}|R|^{\frac{1}{2}}|B| \sim \ tN^{-l-\frac{3}{2}}. \tag{4.11}$$

On the other hand, using (4.2), the Fubini's theorem, $(3.14)_R$, (4.4), (4.8), Young's convolution inequality and (4.10), we get that

$$\begin{aligned}\|I^-_{A,B}\|_{L^2} &= \left\|\int_{\mathbb{R}^d} \tfrac{\langle\xi\rangle^k}{\langle\xi_1\rangle^k\langle\xi_2\rangle^l}\tfrac{\sin(\sigma_- t)}{\sigma_-}\chi_A(\xi_1)\chi_B(\xi_2)d\xi_1\right\|_{L^2_\xi} \ \lesssim \ \left\|\frac{1}{N^l}\frac{1}{N}\ \chi_A * \chi_B\right\|_{L^2}\\ &\leq \ \frac{|A||B|^{\frac{1}{2}}}{N^{l+1}} \sim \ N^{-l-\frac{5}{2}}.\end{aligned} \tag{4.12}$$

Finally, combining (4.1), (4.11), (4.12) and (4.10) we conclude that

$$tN^{-l-\frac{3}{2}} - N^{-l-\frac{5}{2}} \lesssim N^{-1}, \qquad \forall N \in \mathbb{N}.$$

Hence $l \geq -1/2$ when the *flow mapping* (1.3) is C^2 at the origin.

Now we will show that $l \leq 2k - 1/2$ dealing with (3.7). Similarly to the manner that we obtained (3.9), using now $\Phi_0 = (\varphi, 0, 0)$ and $\Phi_1 = (\upsilon, 0, 0)$ in (3.7) with $\varphi, \upsilon \in \mathscr{S}(\mathbb{R}^d)$, we obtain

$$\Bigg\| \langle \xi \rangle^l \int_0^t \frac{e^{i(t-s)|\xi|} - e^{-i(t-s)|\xi|}}{2i|\xi|} |\xi|^2 \int_{\mathbb{R}^d} \Big\{ e^{-is|\xi_1|^2} \widehat{\varphi}(\xi_1) e^{is|\xi_2|^2} \overline{\widehat{\upsilon}(-\xi_2)} + e^{is|\xi_1|^2} \overline{\widehat{\varphi}(-\xi_1)} e^{-is|\xi_2|^2} \widehat{\upsilon}(\xi_2) \Big\} d\xi_1 ds \Bigg\|_{L^2_\xi} \lesssim \|\varphi\|_{H^k} \|\upsilon\|_{H^l}.$$

Similarly to (3.9) and (3.11), from the last estimate follows that, for bounded subsets $A, B \subset \mathbb{R}^d$, we have

$$\Bigg\| \int_0^t \int \frac{\langle \xi \rangle^l |\xi|}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^k} \left(e^{i(t-s)|\xi|} - e^{-i(t-s)|\xi|} \right) \Big(e^{-is(|\xi_1|^2 - |\xi_2|^2)} \chi_A(\xi_1) \chi_{-B}(\xi_2) + e^{is(|\xi_1|^2 - |\xi_2|^2)} \chi_{-A}(\xi_1) \chi_B(\xi_2) \Big) d\xi_1 ds \Bigg\|_{L^2_\xi} \lesssim |A|^{\frac{1}{2}} |B|^{\frac{1}{2}}.$$

So, under the additional assumption that the sets $(A + (-B))$ and $((-A) + B)$ are disjoint[6], the last estimate can be used to obtain

$$\begin{aligned} \|J^+_{A,B}(\xi)\|_{L^2_\xi} - \|J^-_{A,B}(\xi)\|_{L^2_\xi} &\leq \\ &\leq \Bigg\| \int_0^t \int_{\mathbb{R}^d} \frac{\langle \xi \rangle^l |\xi|}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^k} \left(e^{it|\xi| - is\zeta_+} - e^{-it|\xi| - is\zeta_-} \right) \chi_A(\xi_1) \chi_{-B}(\xi_2) d\xi_1 ds \Bigg\|_{L^2_\xi} \\ &\lesssim |A|^{\frac{1}{2}} |B|^{\frac{1}{2}}, \end{aligned} \tag{4.13}$$

where ζ_+ and ζ_- are *the algebraic relations* associated to (3.7) given by

$$\zeta_\pm := |\xi_1|^2 - |\xi_2|^2 \pm |\xi| = \xi^1(\xi_1^1 - \xi_2^1 \pm 1) \pm (|\xi| - \xi^1) + \sum_{j=2}^d \xi^j (\xi_1^j - \xi_2^j) \tag{4.14}$$

and

$$J^\pm_{A,B}(\xi) := |\xi| \int_0^t \int_{\mathbb{R}^d} \frac{\langle \xi \rangle^l}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^k} e^{-is\zeta_\pm} \chi_A(\xi_1) \chi_{-B}(\xi_2) d\xi_1 ds.$$

[6]Since $\chi_X(\xi_1)\chi_Y(\xi_2) = \chi_{X+Y}(\xi = \xi_1 + \xi_2)\chi_X(\xi_1)\chi_Y(\xi_2)$ and $\|f\chi_Z + g\chi_W\|^2_{L^2} = \|f\chi_Z\|^2_{L^2} + \|g\chi_W\|^2_{L^2} \geq \|f\chi_Z\|^2_{L^2}$ when $Z \cap W = \emptyset$.

Now, in view of (4.14), we choose the sets A and B. So, for $N \in \mathbb{N}$ and $0 < \delta < \min\{\frac{1}{7t}, 1\}$, we define

$$A = A_N \ := \ \left[N \, , N + \tfrac{\delta}{N}\right] \times \left[0 \, , \tfrac{\delta}{d-1}\right]^{d-1}$$

and

$$B = B_N \ := \ \left[-N-1 \, , \, -N-1+\tfrac{\delta}{2N}\right] \times \left[-\tfrac{\delta}{2(d-1)} \, , 0\right]^{d-1}.$$

Then $(A+(-B)) \cap ((-A)+B) = \emptyset$ and $\langle \xi_1 \rangle \sim \langle \xi_2 \rangle \sim \langle \xi_1 + \xi_2 \rangle \sim N$, for $(\xi_1, \xi_2) \in A_N \times B_N$. Moreover, following the procedure used in (4.3)-(4.8), one can verify that $\zeta_+ \in (-\delta \, , \, 7\delta)$ and $\zeta_- \in (-7N \, , \, -N)$. Therefore, we have

$$|J^+_{A,B}(\xi)| \ \gtrsim \ tN^{l-2k+1} \chi_A * \chi_B(\xi). \tag{4.15}$$

Consider the set

$$R = R_N \ := \ \left[2N+1 \, , 2N+1+\tfrac{\delta}{2N}\right] \times \left[\tfrac{\delta}{2(d-1)} \, , \, \tfrac{\delta}{(d-1)}\right]^{d-1}$$

and note that $R - (-B) \subset A$ and $|R| \sim |A| \sim |B| \sim N^{-1}$. Then, using (4.15) and Lemma 3.1, we obtain that

$$\|J^+_{A,B}\|_{L^2} \ \gtrsim \ tN^{l-2k+1} |R|^{\frac{1}{2}} |B| \sim \ tN^{l-2k-\frac{1}{2}}. \tag{4.16}$$

On the other hand, similarly to (4.12), we get that

$$\|J^-_{A,B}\|_{L^2} \ = \ \left\| |\xi| \int_{\mathbb{R}^d} \tfrac{\langle \xi \rangle^l}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^k} \tfrac{(e^{-it\zeta_-}-1)}{-i\zeta_-} \chi_A(\xi_1) \chi_{-B}(\xi_2) d\xi_1 \right\|_{L^2_\xi} \ \lesssim \ N^{l-2k-\frac{3}{2}}. \tag{4.17}$$

Finally, combining (4.13), (4.16) and (4.17) we conclude that

$$tN^{l-2k-\frac{1}{2}} - N^{l-2k-\frac{3}{2}} \ \lesssim \ |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} \ \sim \ N^{-1}, \qquad \forall N \in \mathbb{N}.$$

Hence $l \leq 2k - 1/2$ when the *flow mapping* (1.3) is C^2 at the origin. □

5 PROOF OF THEOREM ??

Assume that the *solution mapping* (1.2) is C^2 at the origin. Employing the same procedure that yields (3.11) from (3.4), one can conclude, from (3.5), the following estimate for bounded subsets $A, B \subset \mathbb{R}^d$

$$\sup_{t \in [0,T]} \left\| \int_0^t \int_{\mathbb{R}^d} \tfrac{\langle \xi \rangle^k}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^l} \cos(s|\xi|^2 - s|\xi_1|^2) \cos(s|\xi_2|) \chi_A(\xi_1) \chi_B(\xi_2) d\xi_1 ds \right\|_{L^2_\xi} \lesssim |A| + |B|. \tag{5.1}$$

For $N \in \mathbb{N}$, defining $\vec{N} := (N, 0, \ldots, 0) \in \mathbb{R}^d$,

$$A_N := \{\xi_1 \in \mathbb{R}^d : |\xi_1| < 1/2\}, \qquad B_N := \{\xi_2 \in \mathbb{R}^d : |\xi_2 - \vec{N}| < 1/4\},$$

$$R_N := \{\xi \in \mathbb{R}^d : |\xi - \vec{N}| < 1/4\} \qquad \text{and} \qquad t_N := \frac{1}{4N^2} \cdot \frac{T}{1+T} \,,$$

then $R_N - B_N \subset A_N$, $t_N \in (0,T)$ and, for $(\xi_1, \xi_2) \in A_N \times B_N$, we have

$$\tfrac{\langle \xi \rangle^k}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^l} \sim N^{k-l} \qquad \text{and} \qquad \cos(s|\xi|^2 - s|\xi_1|^2)\cos(s|\xi_2|) > 1/4\,,\ \ \forall s \in [0, t_N].$$

Thus, from Lemma 3.1 and (5.1) yields

$$t_N |R_N|^{\frac{1}{2}} |B_N| N^{k-l} \lesssim \left\| N^{k-l}\, \chi_{A_N} * \chi_{B_N}(\xi) \int_0^{t_N} ds \right\|_{L^2} \lesssim |A_N| + |B_N|, \quad \forall N \in \mathbb{N}. \tag{5.2}$$

Note that $|A_N|$, $|B_N|$ and $|R_N|$ are independent of N. Hence $l \geq k-2$ when the *solution mapping* (1.2) is C^2.

Now we will show that $l \leq k+1$. From (3.5) follows that (3.8) holds uniformly for $t \in [0,T]$. Let $A, B \subset \mathbb{R}^d$ symmetric sets. By using, in (3.8), $\Phi_0 = (\varphi, 0, 0)$ and $\Phi_1 = (\upsilon, 0, 0)$ such that $\varphi, \upsilon \in \mathscr{S}(\mathbb{R}^d)$, $\langle \cdot \rangle^k \widehat{\varphi} \sim \chi_A$ and $\langle \cdot \rangle^k \widehat{\upsilon} \sim \chi_B$ we conclude the following estimate for bounded subsets $A, B \subset \mathbb{R}^d$

$$\sup_{t \in [0,T]} \left\| \int_0^t \cos((t-s)|\xi|)|\xi|^2 \int_{\mathbb{R}^d} \tfrac{\langle \xi \rangle^{l-1}}{\langle \xi_1 \rangle^k \langle \xi_2 \rangle^k} \cos(|\xi_1|^2 s - |\xi_2|^2 s)\, \chi_A(\xi_1) \chi_B(\xi_2) d\xi_1 ds \right\|_{L^2_\xi} \lesssim |A|^{\frac{1}{2}} |B|^{\frac{1}{2}}. \tag{5.3}$$

For $N \in \mathbb{N}$, define

$$A_N := \{\xi_1 \in \mathbb{R}^d : |\xi_1 - \vec{N}| < 1/2\} \cup \{\xi_1 \in \mathbb{R}^d : |\xi_1 + \vec{N}| < 1/2\},$$

$$B_N := \{\xi_2 \in \mathbb{R}^d : |\xi_2| < 1/4\},$$

$$R_N := \{\xi \in \mathbb{R}^d : |\xi - \vec{N}| < 1/4\} \qquad \text{and} \qquad t_N := \frac{1}{4N^2} \cdot \frac{T}{1+T} \,.$$

Note that A_N and B_N are symmetric. Similarly to (5.1)-(5.2), from (5.3) we get the following estimate

$$t_N |R_N|^{\frac{1}{2}} |B_N| N^{l-k+1} \lesssim \left\| N^{l-1-k}\, |\xi|^2\, \chi_{A_N} * \chi_{B_N}(\xi) \int_0^{t_N} ds \right\|_{L^2} \lesssim |A_N|^{\frac{1}{2}} |B_N|^{\frac{1}{2}},$$

for all $N \in \mathbb{N}$. Note that $|A_N|$, $|B_N|$ and $|R_N|$ are independent of N. Hence $l \leq k+1$ when the *solution mapping* (1.2) is C^2. *square*

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