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Continuity of the Blow-Up Time for a Nonlinear Convection in Reaction-Diffusion Equation

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ABSTRACT. In this paper, we consider the following initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta u - g(u).\nabla u + f(u) \quad \text{in} \quad Q \times (0,T), \\ \sigma \partial_t u + \partial_\nu u = 0 \quad \text{on} \quad \partial Q \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in} \quad \overline{Q}, \end{cases}$$

where Q is a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂Q, Δ is the Laplacian, ν is the exterior normal unit vector on ∂Q, $u_0 \in C^2(\overline{Q})$. The dissipative parameter $\sigma \in C^1(\partial Q \times (0,\infty))$ is a bounded function and $u_0(x) > 0, x \in \overline{Q}$. We perturbe the above problem and under some assumptions, we show that the solution of the perturbed form blows up in a finite time and we estimate its blow-up time. We also prove the continuity of the blow-up time with respect to the initial datum.

Keywords: Blow up, nonlinear parabolic equation, dynamical boundary conditions, continuity of the blow up time.

1 INTRODUCTION

Let Q be a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂Q. Consider the following initial-boundary value problem

$$\partial_t u = \Delta u - g(u).\nabla u + f(u) \quad \text{in} \quad Q \times (0,T), \tag{1.1}$$

$$\sigma \partial_t u + \partial_\nu u = 0 \quad \text{on} \quad \partial Q \times (0,T), \tag{1.2}$$

$$u(.,0) = u_0 > 0 \quad \text{in} \quad \overline{Q}, \tag{1.3}$$

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where $g:\mathbb{R}\to\mathbb{R}^N$, $f:\mathbb{R}\to\mathbb{R}$, Q is a bounded domain in $\mathbb{R}^N$ with C^2-boundary ∂Q, ν the exterior unit normal vector on ∂Q.We always assume that the parameters in the equations (1.1)–(1.3) are smooth

$$\sigma \in C^1(\partial Q\times(0,\infty)) \text{ is a bounded function,} \tag{1.4}$$

$$g\in C^1(\mathbb{R},\mathbb{R}^N). \tag{1.5}$$

Typical example of a pair (f,g,u_0) satisfying the assumptions throughout this paper is given by $f(u)=u^p$ with $p>1$, $g(u)=(u^{q_1},\cdots,u^{q_i},\cdots,u^{q_N})$ with $q_i>1$, $i=1,\cdots,N$, $u_0(x)=\frac{2+\cos(\pi\|x\|_\infty)}{4}$ where $\|x\|_\infty=\max\{|x_1|,\cdots,|x_N|\}$.

The problem (1.1)–(1.3) appears in heat conduction theory and in this area blow-up phenomena play an important role. The equation $\partial_t u=\Delta u-g(u).\nabla u+f(u)$ can be interpreted as a heat equation with convective gradient term $g(u).\nabla u$ and f a nonlinear source (see [3] and [8]). A motivation for studying this type of equation is to investigate the effect of the convective gradient term on global existence or nonexistence of solution, and their asymptotic behavior in finite or infinite time. More physical explanations can be found in [14].

For the nonlinear term $f(u)$, our standing assumptions are the following:

(A1) $f:(0,\infty)\to(0,\infty)$ is a C^1 strictly convex, non-decreasing function satisfying $\lim_{s\to\infty}f(s)=\infty$ and $\int_0^\infty \frac{dy}{f(y)}<\infty$.

(A2) There exists a positive constant C_0 such that

$$sf'(H(s))\le C_0 \quad \text{for} \quad s\ge 0, \tag{1.6}$$

where H is the inverse of the function F defined as follows

$$F(s)=\int_s^\infty \frac{dy}{f(y)}.$$

An example of function verifying this assumption is $f(u)=u^p$.

For the initial datum, we make the following hypotheses:

(A3) $u_0\in C^2(\overline{Q})$, $u_0(x)>0$ in $\overline{Q}$, and there exists a positive constant B such that

$$\Delta u_0(x)-g(u_0(x)).\nabla u_0(x)+f(u_0(x))\ge Bf(u_0(x)) \quad \text{in} \quad Q. \tag{1.7}$$

Throughout, we shall assume the dissipativity condition

$$\sigma\ge 0 \quad \text{on} \quad \partial Q\times(0,\infty). \tag{1.8}$$

Here $(0,T)$ is the maximal time interval of existence of the solution u and by a solution we mean the following.

Definition 1.1. *A solution of (1.1)–(1.3) is a function $u(x,t)$ continuous in $\overline{Q}\times[0,T)$, $u(x,t)>0$ in $\overline{Q}\times[0,T)$, and twice continuously differentiable in x and once in t in $Q\times(0,T)$.* The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T}\|u(\cdot,t)\|_\infty=\infty,$$

where $\|u(\cdot,t)\|_\infty=\max_{x\in Q}|u(x,t)|$. In this case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u (see [2] and [6]).

Solutions of nonlinear parabolic equations which blow up in a finite time have interested many authors (see [1]– [25], [4], [6]– [7], and the references cited therein). In particular in [14], the above problem has been considered and existence and uniqueness of a classical solution have been studied (see [14], Theorems 3.3 and 3.4). Under some assumptions, the authors have also shown that the classical solution blows up in a finite time and its blow-up time has been estimated. In this paper, we are interested in the continuity of the blow-up time as a function of the initial datum u_0. More precisely, we consider the following initial-boundary value problem

$$\partial_t\phi=\Delta\phi-g(\phi).\nabla\phi+f(\phi)\quad\text{in}\quad Q\times(0,T_h),\tag{1.9}$$

$$\sigma\partial_t\phi+\partial_\nu\phi=0\quad\text{on}\quad\partial Q\times(0,T_h),\tag{1.10}$$

$$\phi(.,0)=\phi_0>0\quad\text{in}\quad\overline{Q},\tag{1.11}$$

where $\phi_0(x)=u_0(x)+h(x)$, $h\in C^1(\overline{Q})$, $\sigma\partial_t h+\partial_\nu h=0$, $h(x)\geq 0$ in $\overline{Q}$ ($\phi_0(x)=u_0(x)+\frac{1}{1000}$ for example).

Here $(0,T_h)$ is the maximal time interval on which the solution ϕ of (1.9)–(1.11) exists. Definition 1.1 remains valid for the solution ϕ of (1.9)–(1.11) and we state that this solution is sufficiently regular. It is worth noting that the regularity of solutions increases with respect to the regularity of initial data, and one may apply without difficulties the maximum principle (see [14], Section 2, Theorem 2.1.) When T_h is finite, then we say that the solution ϕ of (1.9)–(1.11) blows up in a finite time, and the time T_h is called the blow-up time of the solution ϕ. It follows from the maximum principle that $\phi\geq u$ as long as all of them are defined. We deduce that $T_h\leq T$. In the present paper, we prove that if $\|h\|_\infty$ is small enough, then the solution ϕ of (1.9)-(1.11) blows up in a finite time, and its blow up time T_h goes to T as $\|h\|_\infty$ tends to zero, where T is the blow-up time of the solution u of (1.1)-(1.3) and $\|h\|_\infty=\sup_{x\in\overline{Q}}|h(x)|$ (from convexity of f). Recently, in [16], an analogous study has been done considering the problem (1.1)-(1.3) in the case where $g=0$. Let us notice that in [16], the authors have considered the phenomenon of quenching (we say that a solution quenches in a finite time if it reaches a singular value in a finite time).

The remainder of the paper is organized as follows. In the next section, under some assumptions, we show that the solution ϕ of (1.9)-(1.11) blows up in a finite time and estimate its blow-up time. In the third section, we prove the continuity of the blow-up time.

2 BLOW UP TIME

In this section, under some assumptions, we show that the solution ϕ of (1.9)-(1.11) blows up in a finite time and estimate its blow up time. The result contained in the following theorem proceeds from an idea of Friedman and McLeod in [6]. But before stating the main result, let us recall the result on the maximum principle already proved in [14, Section 2, Theorem 2.1].

Lemma 2.1. *Assume hypotheses (A1), (A3) and* (1.8)-(1.5). *Suppose that σ does not depend on time*

$$\sigma \in C^1(\partial Q). \tag{2.1}$$

Then the solution u of Problem (1.1)–(1.3) *satisfies*

$$u > 0 \quad in \quad \overline{Q} \times (0,T).$$

Now, let us state our result concerning the blow-up time.

Theorem 2.1. *Suppose that there exists a constant $A \in (0,1]$, such that the initial datum at* (1.11) *satisfies*

$$\Delta\phi_0(x) - g(\phi_0(x)).\nabla\phi_0(x) + f(\phi_0(x)) \geq Af(\phi_0(x)) \quad in \quad Q. \tag{2.2}$$

Then, the solution ϕ of (1.9)-(1.11) *blows up in a finite time T_h which obeys the following estimate*

$$T_h \leq \frac{F(\|\phi_0\|_\infty)}{A}.$$

Proof. Since $(0,T_h)$ is the maximal time interval of existence of the solution ϕ, our aim is to show that T_h is finite and satisfies the above inequality. Introduce the function $V(x,t)$ defined as follows

$$V(x,t) = \partial_t\phi(x,t) - Af(\phi(x,t)) \quad \text{in} \quad \overline{Q} \times [0,T_h),$$

where A is the constant defined in (2.2).

A direct calculation reveals that

$$\partial_t V - \Delta V + g(\phi).\nabla V = \partial_t\left(\partial_t\phi - \Delta\phi + g(\phi).\nabla\phi\right) - Af'(\phi)\partial_t\phi \tag{2.3}$$

$$+A\Delta f(\phi) - Ag(\phi).\nabla f(\phi) \quad \text{in} \quad Q \times (0,T_h).$$

A straightforward computation shows that

$$\Delta f(\phi) = f''(\phi)|\nabla\phi|^2 + f'(\phi)\Delta\phi \quad \text{in} \quad Q \times (0,T_h),$$

which implies that

$$\Delta f(\phi) \geq f'(\phi)\Delta\phi \text{ and } \nabla f(\phi) = f'(\phi)\nabla\phi \quad \text{in } Q \times (0,T_h).$$

Using these estimates and (2.3), we arrive at

$$\partial_t V - \Delta V + g(\phi).\nabla V \geq \partial_t\left(\partial_t \phi - \Delta\phi + g(\phi).\nabla\phi\right) \tag{2.4}$$

$$-Af'(\phi)(\partial_t\phi - \Delta\phi + g(\phi).\nabla\phi) \quad \text{in} \quad Q\times(0,T_h).$$

It follows from (1.9), and (2.4) that

$$\partial_t V - \Delta V + g(\phi).\nabla V \geq f'(\phi)\partial_t\phi - Af'(\phi)f(\phi) \quad \text{in} \quad Q\times(0,T_h).$$

Taking into account the expression of V, we find that

$$\partial_t V - \Delta V + g(\phi).\nabla V \geq f'(\phi)V \quad \text{in} \quad Q\times(0,T_h).$$

We also see, according to (1.10), that

$$\sigma\partial_t V = \sigma\partial_t(\partial_t\phi) - A\sigma f'(\phi)\partial_t\phi \quad \text{on} \quad \partial Q\times(0,T_h),$$

which leads to

$$\sigma\partial_t V = \partial_t(-\partial_\nu\phi) + Af'(\phi)(\partial_\nu\phi) \quad \text{on} \quad \partial Q\times(0,T_h),$$

and we have

$$\sigma\partial_t V + \partial_\nu V = 0 \quad \text{on} \quad \partial Q\times(0,T_h).$$

Due to (2.3), we discover that

$$V(x,0) = \Delta\phi_0(x) - g(\phi_0(x)).\nabla\phi_0(x) + f(\phi_0(x)) - Af(\phi_0(x)) \geq 0 \quad \text{in} \quad \overline{Q}.$$

Apply the maximum principle to get

$$V(x,t) \geq 0 \quad \text{in} \quad \overline{Q}\times(0,T_h),$$

or equivalently

$$\partial_t\phi(x,t) - Af(\phi(x,t)) \geq 0 \quad \text{in} \quad Q\times(0,T_h). \tag{2.5}$$

It follows from (2.5) that

$$T_h \leq \frac{1}{A}\int_{\phi(x,0)}^{\phi(x,T_h)} \frac{dy}{f(y)} \quad \text{for} \quad x\in\overline{Q}. \tag{2.6}$$

According to (2.5), we see that ϕ is increasing with respect to the second variable, which implies that $0 \leq \phi(x,0) \leq \phi(x,T_h), x\in\overline{Q}$. Since $f>0$,

$$\int_{\phi(x,0)}^{\phi(x,T_h)} \frac{dy}{f(y)} \leq \int_{\phi(x,0)}^{\infty} \frac{dy}{f(y)} \quad \text{for} \quad x\in Q.$$

We deduce from (2.6) that

$$T_h \leq \frac{F(\|\phi_0\|_\infty)}{A}.$$

Consequently, we deduce that ϕ blows up at the time T_h because the quantity on the right hand side of the above inequality is finite. This ends the proof. □

Remark 2.1. *Let $t \in (0, T_h)$. Integrating the inequality in (2.6) from t to T_h, we get*

$$T_h - t \leq \frac{1}{A} \int_{\phi(x,t)}^{\infty} \frac{dy}{f(y)} \quad \text{for} \quad x \in Q.$$

We deduce that

$$T_h - t \leq \frac{F(\|\phi(\cdot,t)\|_\infty)}{A} \quad \text{for} \quad t \in (0, T_h).$$

Remark 2.2. *In view of the condition (1.7) and reasoning as in the proof of Theorem 2.1, it is not hard to see that there exists a positive constant B such that $\|u_0(t)\|_\infty \geq H(B(T-t))$ for $t \in (0,T)$.*
We also need the following result which shows an upper bound of $\|\phi(\cdot,t)\|_\infty$ for $t \in (0,T)$.

Theorem 2.2. *Let ϕ be solution of (1.9)–(1.11). Then, the following estimate holds*

$$\|\phi(\cdot,t)\|_\infty \leq H(T-t) \quad \text{for} \quad t \in (0,T).$$

Proof. Let $r(t)$ be the function defined as follows

$$r(t) = \|\phi(\cdot,t)\|_\infty \text{ for } t \in [0, T_h).$$

For each t_1, $t_2 \in [0, T_h)$, there exist x_1, x_2 interior points of Q such that $r(t_1) = \phi(x_1, t_1)$ and $r(t_2) = \phi(x_2, t_2)$. Applying Taylor's expansion, we observe that

$$r(t_2) - r(t_1) \geq \phi(x_2,t_2) - \phi(x_2,t_1) = (t_2 - t_1)\phi_t(x_2,t_2) + o(t_2 - t_1),$$

$$r(t_2) - r(t_1) \leq \phi(x_1,t_2) - \phi(x_1,t_1) = (t_2 - t_1)\phi_t(x_1,t_1) + o(t_2 - t_1),$$

which implies that $r(t)$ is Lipschitz continuous. Further, if $t_2 > t_1$, then

$$\frac{r(t_2) - r(t_1)}{t_2 - t_1} \leq \phi_t(x_1,t_1) + o(1) = \Delta\phi(x_1,t_1) - g(\phi(x_1,t_1)).\nabla(\phi(x_1,t_1))$$

$$+ f(\phi(x_1,t_1)) + o(1).$$

Obviously, it is not hard to see that $\Delta\phi(x_1,t_1) \leq 0$ and $\nabla(\phi(x_1,t_1)) = 0$. Letting $t_2 \to t_1$, we obtain $r'(t) \leq f(r(t))$ for a.e. $t \in (0, T_h)$ or equivalently $\frac{dr}{f(r)} \leq dt$ for a.e $t \in (0, T_h)$. Integrate the above inequality over (t, T_h) to obtain $T_h - t \geq \int_{r(t)}^{\infty} \frac{dy}{f(y)}$ for $t \in (0, T_h)$. Since $r(t) = \|\phi(\cdot,t)\|_\infty$, we arrive at $\|\phi(\cdot,t)\|_\infty \leq H(T_h - t)$ for $t \in (0, T_h)$ and the proof is completed. ☐

Remark 2.3. *According to the last part of the proof of Theorem 2.2, we observe that $T_h \geq \int_{r(0)}^{\infty} \frac{dy}{f(y)}$. Thus, we have a lower bound of the blow-up time of the solution ϕ of (1.9)–(1.11). In the same way, it is not hard to see that $\int_{\|u_0\|_\infty}^{\infty} \frac{dy}{f(y)}$ is a lower bound of the blow-up time of the solution u of (1.1)–(1.3).*

3 CONTINUITY OF THE BLOW-UP TIME

This section is devoted to our main result. Under some assumptions, we show that the solution ϕ of (1.9)-(1.11) blows up in a finite time and its blow-up time goes to that of the solution u of (1.1)-(1.3) when h tends to zero in L^∞ norm.

We also provide an upper bound of $|T_h - T|$ in terms of $\|\phi_0 - u_0\|_\infty$. Our result regarding the continuity of the blow-up time is stated in the following theorem.

Theorem 3.1. *Suppose that the problem (1.1)–(1.3) has a solution u which blows up at the time T. Then, under the assumptions of Theorem 2.1, the solution ϕ of (1.9)–(1.11) blows up in a finite time T_h, and there exist positive constants β and λ such that the following estimate holds*

$$|T_h - T| \leq \beta \|\phi_0 - u_0\|_\infty^\lambda,$$

when h goes to zero in L^∞ norm.

Proof. We know from Theorem 2.1 that the solution ϕ blows up in a finite time T_h. Now, to achieve our objective, it remains to prove the above estimate. We begin by proving that $T_h \leq T$. In order to obtain this result, we proceed as follows. Since $\phi_0(x) \geq u_0(x)$ in Q, we know from the maximum principle that $\phi \geq u$ for $t \in (0, T^*)$ with $T^* = \min\{T, T_h\}$. In fact, since ϕ is bigger than u, ϕ reaches ∞ before u, this implies that $T_h \leq T$, and consequently, we have $T - T_h = |T_h - T|$. In order to show the remaining part of the proof, we proceed by introducing the error function $\varepsilon(x,t)$ defined as follows

$$\varepsilon(x,t) = \phi(x,t) - u(x,t) \quad \text{in} \quad \overline{Q} \times [0, T).$$

Let t_0 be any positive quantity satisfying $t_0 < T$. By the mean value theorem, we have

$$\begin{aligned}
\partial_t \varepsilon - \Delta \varepsilon &= g(u) \cdot \nabla u - g(\phi) \cdot \nabla \phi + f(\phi) - f(u) \\
&= -g(\phi) \cdot \nabla(\phi - u) + (g(u) - g(\phi)) \cdot \nabla u + f(\phi) - f(u) \\
&= \varepsilon g'(\theta_1) \cdot \nabla u - g(\phi) \cdot \nabla \varepsilon + f'(\theta)\varepsilon,
\end{aligned}$$

where θ and θ_1 are intermediate values between u and ϕ.

Since g is a smooth function and Q is bounded with smooth boundary ∂Q, there exists a positive constant M such that $-g(\phi) \cdot \nabla \varepsilon + \varepsilon g'(\theta_1) \cdot \nabla u \leq M |\nabla \varepsilon|$, and we obtain

$$\partial_t \varepsilon - \Delta \varepsilon \leq M. \mid \nabla \varepsilon \mid + f'(\theta)\varepsilon \quad \text{in} \quad Q \times (0, t_0),$$

$$\sigma \partial_t \varepsilon + \partial_\nu \varepsilon = 0 \quad \text{on} \quad \partial Q \times (0, t_0),$$

$$\varepsilon(x, 0) = \phi_0(x) - u_0(x) \quad \text{in} \quad \overline{Q}.$$

Due to the fact that $\phi(x,t) \geq u(x,t)$ in $Q \times (0, t_0)$, and making use of Theorem 2.2, it is easy to check that

$$\theta(x,t) \leq \|\phi(\cdot, t)\|_\infty \leq H(T - t) \quad \text{in} \quad Q \times (0, t_0), \tag{3.1}$$

and since f' is increasing by convexity of f, we have

$$\partial_t \varepsilon \leq \Delta\varepsilon + M. \mid \nabla\varepsilon \mid + f'(H(T-t))\varepsilon \quad \text{in} \quad Q\times(0,t_0).$$

In view of the condition (1.6), we observe that there exists a positive constant C_0 such that

$$\partial_t \varepsilon \leq \Delta\varepsilon + M. \mid \nabla\varepsilon \mid + \frac{C_0}{T-t}\varepsilon \quad \text{in} \quad Q\times(0,t_0).$$

Let $W(t)$ be the solution of the following ODE

$$W'(t) = \frac{C_0 W(t)}{T-t} \quad \text{for} \quad t\in(0,t_0), \quad W(0) = \|\phi_0 - u_0\|_\infty.$$

When we solve the above ODE, we observe that its solution $W(t)$ is given explicitly by

$$W(t) = T^{C_0}\|\phi_0 - u_0\|_\infty (T-t)^{-C_0} \quad \text{for} \quad t\in[0,t_0).$$

On the other hand, an application of the maximum principle renders

$$\varepsilon(x,t) \leq W(t) = C_1\|\phi_0 - u_0\|_\infty (T-t)^{-C_0} \quad \text{in} \quad Q\times[0,t_0),$$

where $C_1 = T^{C_0}$. Fix a a positive constant and let $t_1 \in (0,T)$ be a time such that $\|\varepsilon(\cdot,t_1)\|_\infty \leq C_1\|\phi_0 - u_0\|_\infty (T-t_1)^{-C_0} = a$ for h small enough. This implies that

$$T - t_1 = \left(\frac{C_1\|\phi_0 - u_0\|_\infty}{a}\right)^{\frac{1}{C_0}}. \tag{3.2}$$

Making use of Remark 2.1 and the triangle inequality, it is easy to see that

$$|T_h - t_1| \leq \frac{F(\|\phi(\cdot,t_1)\|_\infty)}{A} \leq \frac{F(\|\phi(\cdot,t_1)\|_\infty + \|\varepsilon(\cdot,t_1)\|_\infty)}{A}.$$

Since $\|\varepsilon(\cdot,t_1)\|_\infty \leq a$ and due to the fact that the function $F:[0,\infty)\to[0,\infty)$ is non-decreasing, we infer from Theorem 2.2 that

$$|T_h - t_1| \leq \frac{1}{A}F(H(T-t_1)+a). \tag{3.3}$$

Having in mind that H is the inverse of F, we deduce that $H:[0,\infty)\to[0,\infty)$ is also non-decreasing. We recall that $\lim_{s\to\infty}F(s)=\infty$, which implies that $\lim_{s\to\infty}H(s)=\infty$. Introduce the function ψ defined as follows

$$\psi(x) = H(x(T-t_1)), \quad x\in[0,\infty).$$

It is clear that $\psi(x)$ is non-decreasing for $x\in[0,\infty)$. In addition $\lim_{x\to\infty}\psi(x)=\infty$. According to the fact that $\psi(1)+a$ belongs to $(0,\infty)$, we conclude that there exists a positive constant C_2 such that $\psi(1)+a \leq \psi(C_2)$, which implies that $H(T-t_1)+a \leq H(C_2(T-t_1))$. Recalling that $F(x)$ is non-decreasing for $x\in[0,\infty)$, we deduce that $\frac{1}{A}F(H(T-t_1)+a) \leq \frac{1}{A}F(H(C_2(T-t_1)))$. Exploiting the above inequality and (3.3), we find that

$$|T_h - t_1| \leq \frac{1}{A}F(H(C_2(T-t_1))) = \frac{C_2}{A}|T-t_1|. \tag{3.4}$$

We deduce from (3.4) and the triangle inequality that

$$|T - T_h| \leq |T - t_1| + |T_h - t_1| \leq |T - t_1| + \frac{C_3}{A}|T - t_1|,$$

which leads us, using equality (3.2) to

$$|T - T_h| \leq (1 + \frac{C_2}{A}) \left(\frac{C_1 \|\phi_0 - u_0\|_\infty}{a} \right)^{\frac{1}{C_0}}.$$

Which ends the proof. □

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