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Positivity and equilibrium in a fractional SIR model with Mittag-Leffler memory

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ABSTRACT. We present two slightly different constructions of a SIR model in which both the time taken to remove the individual from the infectious compartment and the infectivity have a memory according to Mittag-Leffler distributions. The second construction clearly points out where the proposed generalizations are acting, starting from the classic SIR model. Using integrodifferential techniques, we state and demonstrate novel analytically results on positivity, monotonicity in limiting case, and equilibrium points. The results are also verified numerically.

Keywords: Fractional SIR model, Mittag-Leffler functions, positivity, integrodifferential equations, equilibrium.

1 INTRODUCTION

Fractional differential equations are extensively applied in memory-dependent mathematical models. In fact, the Fractional Calculus includes the possibility of considering the memory of the studied phenomenon, that is, the dependence on previous stages, once its operators are non local. Most often, a fractional model is designed in an *ad hoc* conception: a classical model is generalized by allowing the integer order of the derivative to be arbitrary. However, this practice can lead to some physical misinterpretation, and previous works study manners to construct fractional models with biological meaning [1, 2, 3, 17].

This work aims to complete previous discussions of fractional SIR models with Mittag-Leffler memory, presenting the detailed proofs of our results in [16]. To this aim, in Subsection 2.1, we

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present an overview of the work [3], a recent and unusual SIR model constructed by Continuous Time Random Walks (CTRW). Both the time of removal of the individual from the infectious compartment and the infectivity are age-dependent and follow a Mittag-Leffler distribution, causing the Riemann-Liouville fractional derivative to arise throughout Laplace Transform techniques. In Subsection 2.2, we show how to obtain from the theory presented in [15] the same fractional model constructed with CTRW. It shows clearly where the generalizations proposed are acting, from the viewpoint of the classical SIR model. Then, we propose and demonstrate new results about positivity, monotonicity (Section 3), and equilibrium points (Section 4).

MATLAB simulations with codes proposed in [17] and other references verify the theory numerically. We use basic definitions and results of Fractional Calculus from classic references [6, 12, 14, 20, 22, 23]. We denote $I^{1-\alpha}$ the Riemann-Liouville integral and $D^{1-\alpha}$ the Riemann-Liouville derivative operators with initial point zero.

2 THE MODEL

The construction proposed in [3] is studied in details in [17, 19] and correlated references. Here, we present a brief overview, for completeness.

2.1 An overview

The main idea is that, if there are $S(t)$ susceptible people at time t in a population of size N, then the expected number of new infections per infected individual is given by $\sigma(t,t')\frac{S(t)}{N}\Delta T$ in a step time ΔT. Here, $\sigma(t,t')$ is the transmission function of an individual first infected at time $t' < t$.

The survival function $\Phi(t,t')$ gives the probability that a person first infected at time $t' < t$ stays infected at time t. Considering that there is no disease before $t = 0$, the flow of individuals into infectious compartment I in time t is recursively given by

$$q^+(I,t) = \int_0^t \sigma(t,t')\frac{S(t)}{N}\Phi(t,t')q^+(I,t')dt' + i_0\sigma(t,0)\frac{S(t)}{N}\Phi(t,0), \tag{2.1}$$

where i_0 is the initial condition: the quantity of infectious that emerged in time 0.

The transmission function $\sigma(t,t')$ is dependent of both the present t, due, e.g., to cultural or containment measures, and the age of infection $t-t'$. The dependence of t is expressed by an extrinsic infectivity ω, independent of the individual. On the other hand, there is an intrinsic infectivity ρ dependent of age. So,

$$\sigma(t,t') = \omega(t)\rho(t-t'). \tag{2.2}$$

For the survival function, it is assumed that natural death and recuperation/death by disease of an infected individual are independent. Then,

$$\Phi(t,t') = \phi(t-t')\theta(t,t'), \tag{2.3}$$

where $\phi(t-t')$ is the probability that an individual first infected in t' is still with the disease at time t. Furthermore, $\theta(t,t')$ is the probability that an individual first infected in t' has not yet died a natural death by time t, being given by

$$\theta(t,t') = e^{-\int_{t'}^{t}\gamma(u)du}, \tag{2.4}$$

where γ is the vital rate.

Individuals in compartment I at time t must have entered this compartment at some previous time and remained there until t. Therefore, we can express the number of infected individuals as follows:

$$I(t) = \Phi(t,0)i_0 + \int_0^t \Phi(t,t')q^+(I,t')dt'. \tag{2.5}$$

Writing $\psi(t) = -d\phi(t)/dt$, we derive Eq. (2.5) through Leibniz's Rule, obtaining

$$\begin{aligned}\frac{dI(t)}{dt} = \omega(t)\frac{S(t)}{N}\left(\int_0^t \rho(t-t')\Phi(t,t')q^+(I,t')dt' + \rho(t)\Phi(t,0)i_0\right)\\ - \int_0^t \psi(t-t')\theta(t,t')q^+(I,t')dt' - \psi(t)\theta(t,0)i_0 - \gamma(t)I(t).\end{aligned} \tag{2.6}$$

The dependency on $q^+(I,t')$ is removed by defining memory kernels for infectivity and recovery. Henceforth we consider $i_0 = 1$ for simplicity. Eq. (2.5) can be rewritten as

$$\frac{I(t)}{\theta(t,0)} - \phi(t) = \int_0^t \phi(t-t')\frac{q^+(I,t')}{\theta(t',0)}dt'. \tag{2.7}$$

Using the Laplace transform in the first integral of (2.6) we can write

$$\mathscr{L}\{\rho(t)\phi(t)\}\mathscr{L}\left\{\frac{q^+(I,t)}{\theta(t,0)}\right\} = \mathscr{L}\left\{\int_0^t K_I(t-t')\left(\frac{I(t')}{\theta(t',0)}\right)dt' - \rho(t)\phi(t)\right\}, \tag{2.8}$$

where it is defined the infectivity memory kernel

$$K_I(t) = \mathscr{L}^{-1}\left\{\frac{\mathscr{L}\{\rho(t)\phi(t)\}}{\mathscr{L}\{\phi(t)\}}\right\}. \tag{2.9}$$

In the same way, based in the second integral of Eq. (2.6),

$$\mathscr{L}\{\psi(t)\}\mathscr{L}\left\{\frac{q^+(I,t)}{\theta(t,0)}\right\} = \mathscr{L}\left\{\int_0^t K_R(t-t')\left(\frac{I(t')}{\theta(t',0)}\right)dt' - \psi(t)\right\}, \tag{2.10}$$

where it is defined the recovery memory kernel by

$$K_R(t) = \mathscr{L}^{-1}\left\{\frac{\mathscr{L}\{\psi(t)\}}{\mathscr{L}\{\phi(t)\}}\right\}. \tag{2.11}$$

Fractional derivatives are incorporated into the model by choosing $\psi(t)$ with potential law and $\rho(t)$ related to the choice of $\psi(t)$:

$$\psi(t) = \frac{t^{\alpha-1}}{\tau^{\alpha}} E_{\alpha,\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right), \tag{2.12}$$

for $0 < \alpha \leq 1$, where τ is a scale parameter. The corresponding survival function is

$$\phi(t) = E_{\alpha,1}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right). \tag{2.13}$$

We can calculate the Laplace transform of the removal memory kernel defined by the Mittag-Leffler function, obtaining:

$$\mathscr{L}\{K_R(t)\} = \frac{\mathscr{L}\{\psi(t)\}}{\mathscr{L}\{\phi(t)\}} = s^{1-\alpha}\tau^{-\alpha}. \tag{2.14}$$

The Laplace transform of the kernel implies that a convolution with the recovery memory kernel can be written as

$$\int_0^t K_R(t-t')\frac{I(t')}{\theta(t',0)}dt' = \tau^{-\alpha} D^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right) \tag{2.15}$$

Another fractional derivative is incorporated into the infectivity memory kernel by considering

$$\rho(t) = \frac{1}{\phi(t)}\frac{t^{\beta-1}}{\tau^{\beta}} E_{\alpha,\beta}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right). \tag{2.16}$$

Since $\rho(t) \geq 0$ is necessary, we take $0 < \alpha \leq \beta \leq 1$. Using Eq. (2.16), we obtain the Laplace transform of the infectiousness kernel:

$$\int_0^t K_I(t-t')\frac{I(t')}{\theta(t',0)}dt' = \tau^{-\beta} D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right). \tag{2.17}$$

Substituting Eqs. (2.15) and (2.17) into the master equation Eq.(2.6) and rewriting, we obtain a fractional SIR model:

$$S'(t) = \gamma(t)N - \frac{\omega(t)S(t)\theta(t,0)}{N\tau^{\beta}} D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right) - \gamma(t)S(t), \tag{2.18}$$

$$I'(t) = \frac{\omega(t)S(t)\theta(t,0)}{N\tau^{\beta}} D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right) - \frac{\theta(t,0)}{\tau^{\alpha}} D^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right) - \gamma(t)I(t), \tag{2.19}$$

$$R'(t) = \frac{\theta(t,0)}{\tau^{\alpha}} D^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right) - \gamma(t)R(t). \tag{2.20}$$

Once there is no disease before time 0, the initial condition must be in the form $S(0) = N - i_0, I(0) = i_0, R(0) = 0$.

2.2 Another approach for constructing a fractional SIR model

The function $\psi(t)$ can be related with the continuous random variable X that gives the individual's waiting time in infectious compartment despite the vital dynamics. To see this, note that the cumulative distribution of X is $F(t) = P(X \leq t) = 1 - \phi(t)$. Thus, the probability density function of X is $\psi(t) = -d\phi(t)/dt$. The probability distribution F is a Mittag-Leffler distribution $F(t;\alpha,\tau) = 1 - E_\alpha\left(-\left(\frac{t}{\tau}\right)^\alpha\right)$. If $\alpha < 1$, the expectation is infinite. This can represent, for instance, cases in which long waiting times are not really improbable. However, if $\alpha = 1$, it represents an exponential distribution. For waiting times exponentially distributed, the first moment of the random variable X exists, and τ is exactly the mean waiting time in infectious compartment despite the vital dynamics. It is worth to note that, if $\gamma(t) \equiv \gamma > 0$, the effective mean waiting time in infectious compartment, described by both the effects of recovery and natural death, is finite and given by $\gamma^{\alpha-1}/(\gamma^\alpha + \tau^{-\alpha})$ for any $\alpha \in (0,1]$ [15]. In this Section, we show how to obtain from the presented theory in [15] the same fractional model constructed with CTRW in the previous Section.

The compartmental SIR (Susceptible-Infected-Removed) model was introduced by Kermack and McKendrick in 1927 [11]. With constants rates of vital dynamics (γ), infectivity (σ) and mean removal time (τ), the SIR model can be written as

$$\begin{cases} S'(t) = \gamma N - \sigma\dfrac{S(t)}{N}I(t) - \gamma S(t), \\ I'(t) = \sigma\dfrac{S(t)}{N}I(t) - \frac{1}{\tau}I(t) - \gamma I(t), \\ R'(t) = \frac{1}{\tau}I(t) - \gamma R(t). \end{cases} \tag{2.21}$$

From the second equation, we can write

$$I = \int_0^t e^{-\gamma(t-t')}e^{-(t-t')/\tau}\sigma\frac{S(t')}{N}I(t')dt' + e^{-\gamma t}e^{-t/\tau}i_0. \tag{2.22}$$

We generalize $e^{-t/\tau}$, which indicates an exponentially distributed waiting time in the infectious compartment, with a Mittag-Leffler probability $E_\alpha\left(-\left(\frac{t}{\tau}\right)^\alpha\right)$.

This generalization constructs a fractional SIR model with general remotion time as in [2]. However, as seen in previous Subsection, in [3] there is also a generalization in the transmission function: in fact, the infectivity σ is seen as a function with an extrinsic term $\omega(t)$, depending on the present time (for instance, cultural and political measures), and an intrinsic term $\rho(t-t')$, depending on the age of infection:

$$\sigma(t,t') = \omega(t)\rho(t-t'). \tag{2.23}$$

If there is no memory, i.e., if the force of infectivity is the same along the days of infection, so ρ is a constant.

Note that a people that becomes infectious in a time t' is still infectious in t if:

- It is alive (probability $e^{-\gamma(t-t')}$), and
- It is not removed (probability $E_\alpha\left(-\left(\frac{(t-t')}{\tau}\right)^\alpha\right)$).

In this case, its infectivity is $\omega(t)\rho(t-t')$. So, the flux of new infectious is given recursively by

$$\begin{aligned} q^+(t) =& \frac{S(t)}{N}\omega(t)\int_0^t e^{-\gamma(t-t')}E_\alpha\left(-\left(\frac{(t-t')}{\tau}\right)^\alpha\right)\rho(t-t')q^+(t')dt' \\ &+ \frac{S(t)}{N}\omega(t)e^{-\gamma t}E_\alpha\left(-\left(\frac{t}{\tau}\right)^\alpha\right)\rho(t)i_0. \end{aligned} \tag{2.24}$$

Aiming to use the Laplace Transform of Mittag-Leffler functions in Eq.(2.24) and considering that the intrinsic infectivity is positive and monotonically decreasing with time, it is proposed

$$\rho(t) = \frac{1}{E_\alpha\left(-\left(\frac{t}{\tau}\right)^\alpha\right)}\frac{t^{\beta-1}}{\tau^\beta}E_{\alpha,\beta}\left(-\left(\frac{t}{\tau}\right)^\alpha\right), \tag{2.25}$$

with $0 < \alpha \leq \beta \leq 1$. Then,

$$\begin{aligned} q^+(t) =& \frac{S(t)}{N}\omega(t)\int_0^t e^{-\gamma(t-t')}\frac{(t-t')^{\beta-1}}{\tau^\beta}E_{\alpha,\beta}\left(-\left(\frac{(t-t')}{\tau}\right)^\alpha\right)q^+(t')dt' \\ &+ \frac{S(t)}{N}\omega(t)e^{-\gamma t}\frac{t^{\beta-1}}{\tau^\beta}E_{\alpha,\beta}\left(-\left(\frac{t}{\tau}\right)^\alpha\right)i_0. \end{aligned} \tag{2.26}$$

Note that

$$\int_0^t e^{-\gamma(t-t')}\frac{(t-t')^{\beta-1}}{\tau^\beta}E_{\alpha,\beta}\left(-\left(\frac{(t-t')}{\tau}\right)^\alpha\right)q^+(t')dt' \tag{2.27}$$

$$= \frac{e^{-\gamma t}}{\tau^\beta}\mathscr{L}^{-1}\left[\mathscr{L}\left[t^{\beta-1}E_{\alpha,\beta}\left(-\left(\frac{t}{\tau}\right)^\alpha\right)\right]\mathscr{L}\left[e^{\gamma t}q^+(t)\right]\right] \tag{2.28}$$

$$= \frac{e^{-\gamma t}}{\tau^\beta}\mathscr{L}^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha+\tau^{-\alpha}}\mathscr{L}\left[e^{\gamma t}q^+(t)\right]\right] \tag{2.29}$$

$$= \frac{e^{-\gamma t}}{\tau^\beta}\mathscr{L}^{-1}\left[s^{1-\beta}\mathscr{L}\left[\int_0^t E_\alpha\left(-\left(\frac{(t-t')}{\tau}\right)^\alpha\right)e^{\gamma t'}q^+(t')dt'\right]\right] \tag{2.30}$$

$$= \frac{e^{-\gamma t}}{\tau^\beta}D^{1-\beta}(e^{\gamma t}Y), \tag{2.31}$$

where

$$Y(t) = \int_0^t E_\alpha\left(-\left(\frac{(t-t')}{\tau}\right)^\alpha\right)e^{-\gamma(t-t')}q^+(t')dt'. \tag{2.32}$$

Note that Y indicates the summation of people that entered in I compartment in each $0 < t' \leq t$ and are still alive and infectious at t. So,

$$\begin{aligned} I(t) &= E_\alpha\left(-\left(\frac{t}{\tau}\right)^\alpha\right)e^{-\gamma t}i_0 + Y(t) \\ &= E_\alpha\left(-\left(\frac{t}{\tau}\right)^\alpha\right)e^{-\gamma t}i_0 + \int_0^t E_\alpha\left(-\left(\frac{(t-t')}{\tau}\right)^\alpha\right)e^{-\gamma(t-t')}q^+(t')dt'. \end{aligned} \tag{2.33}$$

This implies that

$$D^{1-\beta}(e^{\gamma t}Y) = D^{1-\beta}(e^{\gamma t}I) - i_0 D^{1-\beta}\left(E_\alpha\left(-\left(\frac{t}{\tau}\right)^\alpha\right)\right) \tag{2.34}$$

$$= D^{1-\beta}(e^{\gamma t}I) - i_0 t^{\beta-1}E_{\alpha,\beta}\left(-\left(\frac{t}{\tau}\right)^\alpha\right). \tag{2.35}$$

Follows then from (2.26) and (2.35) that

$$q^+(t) = \frac{S(t)\omega(t)e^{-\gamma t}D^{1-\beta}(e^{\gamma t}I)}{N\tau^\beta}. \tag{2.36}$$

Finally, deriving (2.33) by Leibniz rule and using

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty}\frac{k!}{\Gamma(\alpha k+\alpha)}\frac{z^k}{k!} = \sum_{k=0}^{\infty}\frac{(\alpha k+\alpha)k!}{\Gamma(\alpha k+\alpha+1)}\frac{z^k}{k!} = \alpha E^2_{\alpha,1+\alpha}(z), \tag{2.37}$$

we get

$$I'(t) = q^+(t) - \frac{e^{-\gamma t}D^{1-\alpha}(e^{\gamma t}I)}{\tau^\alpha} - \gamma I(t) \tag{2.38}$$

$$= \frac{S(t)\omega(t)e^{-\gamma t}D^{1-\beta}(e^{\gamma t}I)}{N\tau^\beta} - \frac{e^{-\gamma t}D^{1-\alpha}(e^{\gamma t}I)}{\tau^\alpha} - \gamma I(t). \tag{2.39}$$

This is the master equation of the fractional SIR model (2.18)-(2.20) with constant $\gamma(t) \equiv \gamma$:

$$S'(t) = \gamma N - \frac{S(t)\omega(t)e^{-\gamma t}D^{1-\beta}(e^{\gamma t}I)}{N\tau^\beta} - \gamma S(t), \tag{2.40}$$

$$I'(t) = \frac{S(t)\omega(t)e^{-\gamma t}D^{1-\beta}(e^{\gamma t}I)}{N\tau^\beta} - \frac{e^{-\gamma t}D^{1-\alpha}(e^{\gamma t}I)}{\tau^\alpha} - \gamma I(t), \tag{2.41}$$

$$R'(t) = \frac{e^{-\gamma t}D^{1-\alpha}(e^{\gamma t}I)}{\tau^\alpha} - \gamma R(t). \tag{2.42}$$

In next Section, we demonstrate the non-negativity of the solution. Furthermore, we show that, in the limiting case where the vital dynamics is null, the compartments S and R are monotone.

3 MONOTONICITY IN THE LIMITING CASE AND NON-NEGATIVITY

The solution of the fractional SIR model constructed in previous Section is expected to be non-negative, since the compartments reproduce groups of people. In fact, we demonstrate the following result:

Proposition 1.

Let $\gamma(t) \equiv \gamma \geq 0$. Admitting that the solution of the system (2.18)-(2.20) *exists uniquely with initial conditions $S(0) \geq 0, I(0) = i_0 > 0$ and $R(0) = N - S(0) - I(0) \geq 0$, and is continuous for $t > 0$, then:*

- *$S(t), I(t), R(t)$, $D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right)$ and $D^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right)$ are non-negative functions;*
- *$S(t), I(t), R(t) \leq N$ for all t.*

Proof. First, we prove the positivity of $S(t)$ for both $\gamma > 0$ and $\gamma = 0$. Initially, consider $\gamma > 0$. Suppose absurdly that the set $\{t > 0;\ S(t) < 0\}$ is not empty. Let $T > 0$ be the infimum of this set. Once the solution is continuous, we have $S(T) = 0$. From Eq. (2.18), we have $S'(T) = \gamma N > 0$. Thus, there is an interval $(T, T+\varepsilon)$ such that $S'(T) > 0$ and S is increasing in this interval, absurd, since we define T the infimum of the set $\{t > 0;\ S(t) < 0\}$, and so there is a sequence in $\{t > 0;\ S(t) < 0\}$ that converges to T from the right. Therefore, $S(t) > 0$ for all t.

On the other hand, if $\gamma = 0$, we can write

$$S'(t) = -q^+(I,t) = -\int_0^t \sigma(t,t')\frac{S(t)}{N}\Phi(t,t')q^+(I,t')dt' - i_0\sigma(t,0)\frac{S(t)}{N}\Phi(t,0), \tag{3.1}$$

$$\frac{S'(t)}{S(t)} = \frac{1}{N}\int_0^t \sigma(t,t')\Phi(t,t')S'(t')dt' - \frac{1}{N}i_0\sigma(t,0)\Phi(t,0), \tag{3.2}$$

$$\log(|S(s)|) = \frac{1}{N}\int_0^s \left[\int_0^t \sigma(t,t')\Phi(t,t')S'(t')dt' - i_0\sigma(t,0)\Phi(t,0)\right]dt + \log(S(0)), \tag{3.3}$$

$$|S(s)| = e^{f(s)}S(0) = \begin{cases} e^{f(s)}S(0), & S(0) \geq 0 \\ -e^{f(s)}S(0), & S(0) < 0 \end{cases}, \tag{3.4}$$

where

$$f(s) = \frac{1}{N}\int_0^s \left[\int_0^t \sigma(t,t')\Phi(t,t')S'(t')dt' - i_0\sigma(t,0)\Phi(t,0)\right]dt. \tag{3.5}$$

Once $S(0) \geq 0$, Eq.(3.4) implies that $S(t) \geq 0$ for all t.

Given the positivity of $S(t)$, as σ and Φ are positive functions, Eq. (2.1) or Eq. (2.26) gives $q^+(I,t) \geq 0$ for all t. From Eq. (2.5) or Eq. (2.33) it follows directly that $I(t) > 0$ for all t. Now, applying the inverse Laplace Transform to Eq. (2.8), we have

$$\int_0^t K_I(t-t')\frac{I(t')}{\theta(t',0)}dt' = \int_0^t \rho(t-t')\phi(t-t')\frac{q^+(I,t')}{\theta(t',0)}dt' + \rho(t)\phi(t)i_0. \tag{3.6}$$

It follows from the positivity of ρ, ϕ, θ and $q^+(I,t)$ that

$$\int_0^t K_I(t-t')\frac{I(t')}{\theta(t',0)}dt' > 0, \tag{3.7}$$

for all t. Therefore, from Eq. (2.17), we have $D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right) > 0$, for all t. The same can be concluded for $D^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right)$.

Finally, suppose absurdly that the set $\{t > 0;\ R(t) < 0\}$ is not empty. Let $T \geq 0$ be the infimum of this set. Once the solution is continuous and $R(0) \geq 0$, we must have $R(T) = 0$. Then,

$$R'(T) = \frac{\theta(T,0)}{\tau^{\alpha}}\left(D^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right)\right)(T) > 0. \tag{3.8}$$

Thus, there is an interval $(T, T+\varepsilon)$ such that $R'(t) > 0$ and R is increasing in this interval, absurd, since there is a sequence in $\{t > 0;\ R(t) < 0\}$ that converges to T from the right. Therefore, $R(t) \geq 0$ for all t.

Since $S(t) + I(t) + R(t) = N$, it follows from $S(t)$, $I(t)$, $R(t) \geq 0$ that $S(t)$, $I(t)$, $R(t) \leq N$, completing the proof. □

For the limit case in which $\gamma = 0$, we also have the following result:

Proposition 2. *Considering the limit system* (2.18)-(2.20) *for which* $\gamma(t) \equiv 0$*, the functions* $S(t)$ *and* $R(t)$ *are monotone decreasing and increasing, respectively* .

Proof. It follows directly from $S(t)$, $D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right)$ and $D^{1-\alpha}\left(\frac{I(t)}{\theta(t,0)}\right)$ be non-negative that, if $\gamma(t) \equiv 0$, then $S'(t) \leq 0$ and $R'(t) \geq 0$. Thus S and R are monotonically decreasing and increasing, respectively. □

We note that this condition has good epidemiological agreement, but we cannot guarantee this for *ad hoc* SIR fractional models, where the integer order derivative is replaced by the Caputo derivative:

$$\begin{cases} {}^{C}D^{\alpha}_{0+}S(t) = -\sigma\dfrac{S(t)}{N}I(t), \\ {}^{C}D^{\alpha}_{0+}I(t) = \sigma\dfrac{S(t)}{N}I(t) - \frac{1}{\tau}I(t), \\ {}^{C}D^{\alpha}_{0+}R(t) = \frac{1}{\tau}I(t). \end{cases} \tag{3.9}$$

In fact, Figure 1 indicates the solution of the system (3.9). We illustrate the non-monotonicity of the compartments S and R in the model without vital dynamics. This problem is not solved by balancing the units of the parameters initially considered. With the same orders, the corrections would transform them numerically into other constants, but the same in each compartment. On the other hand, in Figure 2 the system (2.18)-(2.20) is numerically solved. We observe that monotonicity remains in the model with arbitrary orders, as observed in Proposition 2, even if $\alpha \neq \beta$. Thus, the oscillations that can appear in this model are due exclusively to vital dynamics. We use a population $N_0 = 10^6$, initial conditions $S(0) = N_0 - 1$, $I(0) = 1$ and $R(0) = 0$ and $dt = 0.1$ for both simulations. Our previous works [13, 18] present a deep discussion about the *ad hoc* fractionalization (3.9).

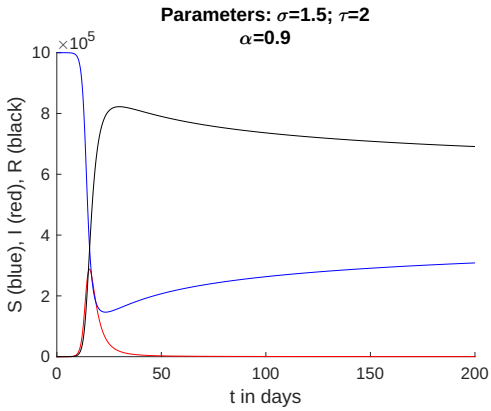

Figure 1: Non-monotonic behavior of S and R in system (3.9).

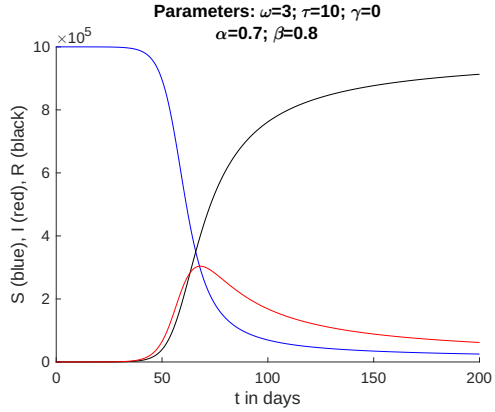

Figure 2: Monotonicity of S and R in (2.18)-(2.20) for $\gamma(t) \equiv 0$.

Next Section deals with equilibrium points. In particular, we study its relationship with the basic reproduction number and approaches to stability analysis.

4 EQUILIBRIUM POINTS AND STABILITY

Classical epidemiological ODEs' model as (2.21) are time invariant [7]. Their local behavior allows the parameters to define the disease dynamics independently of the time in which one begins the modeling. However, this is not valid in model (2.18)-(2.20). Different pasts modify the future dynamics. Its non-local behavior also implies unexpected behavior of the reproduction number $\Re(t)$. The reproduction number in t can be defined as the expected number of individuals infected by a person first infected in time t. The basic reproduction number is this number for $t = 0$, i.e., the average number of secondary infections that occur when an infectious individual is introduced into a completely susceptible population [9]. The definition of the reproduction numbers in the proposed model is non intuitive. We mention that, in [3], it is proposed an integral construction for the basic reproduction number. In [19], we extend the reproduction number proposal to any time t and also propose the S-variable reproduction numbers. These discussions have important implications in equilibrium analysis and peak conditions. For instance, the peak does not occur when $\Re(t) = 1$, nor $\Re^S(t) = 1$ [19].

For this work proposal, we only use the basic reproduction number [3]:

$$\Re_0 = \int_0^\infty \sigma(t)\Phi(t)dt = \frac{\omega\gamma^{\alpha-\beta}}{\tau^\beta\gamma^\alpha + \tau^{\beta-\alpha}}. \tag{4.1}$$

Returning to the equilibrium calculation, we consider $\gamma(t) \equiv \gamma$ constant, so $\theta(t,0) = e^{-\gamma t}$. Taking the limit when $t \to \infty$, we calculate the limits of the form $\lim_{t\to\infty} e^{-\gamma t} D^{1-\alpha}(I(t)e^{\gamma t})$ according to [19]. Assuming $\lim_{t\to\infty} \omega(t) = \omega^*$, we obtain a disease-free state,

$$S^* = N, \quad I^* = 0, \quad R^* = 0, \tag{4.2}$$

and, if $\omega^* > 0$, an endemic state:

$$\begin{aligned} S^* &= \frac{((\tau\gamma)^{\beta-\alpha} + (\tau\gamma)^{\beta})N}{\omega^*}, \qquad I^* = \frac{N(\tau\gamma)^{\alpha}}{1+(\tau\gamma)^{\alpha}} - \frac{N(\tau\gamma)^{\beta}}{\omega^*}, \\ R^* &= \frac{N}{1+(\tau\gamma)^{\alpha}} - \frac{N(\tau\gamma)^{\beta-\alpha}}{\omega^*}. \end{aligned} \tag{4.3}$$

Particularly, when $\omega(t) \equiv \omega$, this result is the same as that obtained in [3]. Note that the viability of the endemic equilibrium (4.3) requires

$$\omega^* > (\tau\gamma)^{\beta-\alpha} + (\tau\gamma)^{\beta}, \tag{4.4}$$

which is related to the basic reproduction number (4.1).

In fact, if $\omega(t) \equiv \omega$, the criterion for the viability given in Eq. (4.4) can be rewritten as $\mathfrak{R}_0 > 1$. Furthermore, the value S^* of the endemic state given in (4.3) is of the form $S^* = N/\mathfrak{R}_0$. Thus, $\mathfrak{R}_0$ has an essential relationship with the final size of the infection.

Until now, we have proven that **if there are asymptotically stable equilibria for the case $\gamma > 0$, then they are given by Eq.**(4.2)**-**(4.3). The limiting case in which $\gamma = 0$ is studied in [16]:

Theorem 1. *If $\omega(t) \equiv \omega$, $\gamma(t) \equiv 0$, $i_0 > 0$ and $\alpha = \beta$ in the system* (2.18)-(2.20)*, then the solution asymptotically approaches equilibrium $(S,I,R)_\infty$, where*

$$R_\infty = N + \frac{N}{\omega} W_0\left(\frac{-S_0\omega e^{-\omega}}{N}\right), \tag{4.5}$$

and

$$S_\infty = S_0 \exp\left(-\frac{\omega}{N}R_\infty\right) = -\frac{N}{\omega}W_0\left(\frac{-S_0\omega e^{-\omega}}{N}\right), \quad I_\infty = N - S_\infty - R_\infty = 0. \tag{4.6}$$

W_0 represents the main branch of Lambert's function W [5], and $S_0 = S(0)$.

Proof. See [16]. □

For $\gamma > 0$, the disease-free state is expected to be an asymptotically stable equilibrium when $\omega^* < (\tau\gamma)^{\beta-\alpha} + (\tau\gamma)^{\beta}$, while the endemic state is expected to be asymptotically stable if $\omega^* > (\tau\gamma)^{\beta-\alpha} + (\tau\gamma)^{\beta}$. These hypothesis are formulated, for constant $\omega(t)$, in [3].

To develop our results, based on [10], we need the integral forms of the infectious and recovered compartments:

$$I(t) = \Phi(0,t)i_0 + \int_0^t \Phi(t,t')\frac{\omega(t')S(t')\theta(t',0)}{N\tau^\beta}\left(D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right)\right)(t')dt', \tag{4.7}$$

$$R(t) = F(t)\theta(t,0)i_0 + \int_0^t F(t-t')\theta(t,t')\frac{\omega(t')S(t')\theta(t',0)}{N\tau^\beta}\left(D^{1-\beta}\left(\frac{I(t)}{\theta(t,0)}\right)\right)(t')dt', \tag{4.8}$$

where $F(t) = 1-\phi(t)$.

The following results are demonstrated:

Theorem 2. *If $\omega(t)$ is bounded with $\lim_{t\to\infty}\omega(t) = \omega^*$, $\gamma(t) \equiv \gamma$ and $\beta = 1$ in the system* (2.18)-(2.20)*, the disease-free equilibrium of Eq.* (4.2) *is globally asymptotically stable if $\omega^* < (\tau\gamma)^{1-\alpha} + (\tau\gamma)$.*

Proof. We define the constant A such that

$$A = \int_0^\infty \frac{\omega^*}{\tau}\Phi(t)dt = \frac{\omega^*}{(\tau\gamma)+(\tau\gamma)^{1-\alpha}} < 1. \tag{4.9}$$

Let $J = \limsup_{t\to\infty} I(t)$ and suppose absurdly that $J > 0$. Thus, it is possible to choose ε small enough such that $2\varepsilon + A(1+\varepsilon)(J+\varepsilon) < J$. Indeed, defining the continuously increasing function $E(t) = 2t + A(1+t)(J+t)$, we have $E(0) = AJ < J$ and $\lim_{t\to\infty} E(t) = \infty$, so there is $\varepsilon_J > 0$ such that $E(\varepsilon_J) = J$. Therefore, we just need to choose $\varepsilon \in (0, \varepsilon_J)$.

We have $\lim_{t\to 0}\Phi(t,0)i_0 = 0$. Furthermore, we know that $\limsup_{t\to\infty} I(t) = J$ and $\lim_{t\to\infty}\omega(t) = \omega^*$. Thus, choosing ε, we can choose a time $t_1 > 0$ sufficiently large such that $\Phi(t,0)i_0 < \varepsilon/2$, $I(t) < J+\varepsilon$ and $\omega(t) < (1+\varepsilon)\omega^*$ for $t > t_1$. Choosing t_1, since $\lim_{t\to 0}\theta(t,0) = 0$, we can choose t_2 large enough such that $Nt_1\theta(t,0)\omega_m/\tau < \varepsilon$ for $t > t_2$, where ω_m is the maximum value reached by $\omega(t)$. So, for $t > t_1 + t_2$, we have $t - t_1 > t_2$ and

$$\begin{aligned} I(t) &= \Phi(t,0)i_0 + \int_0^t \Phi(t,t')\frac{\omega(t')S(t')I(t')}{N\tau}dt' \\ &< \frac{\varepsilon}{2} + \int_0^{t_1}\Phi(t,t')\frac{\omega(t')S(t')I(t')}{N\tau}dt' + \int_{t_1}^t \Phi(t,t')\frac{\omega(t')S(t')I(t')}{N\tau}dt'. \end{aligned} \tag{4.10}$$

Now, we remember that $S(t), I(t) \le N$. In the first integral we use $\Phi(t,0) = \phi(t)\theta(t,0) \le \theta(t,0)$. In the second integral, we use that $I(t) < J+\varepsilon$ and $\omega(t) < (1+\varepsilon)\omega^*$ for $t > t_1$, obtaining

$$I(t) < \frac{\varepsilon}{2} + N\int_0^{t_1}\theta(t,t')\frac{\omega_m}{\tau}dt' + (J+\varepsilon)(1+\varepsilon)\int_{t_1}^t \Phi(t,t')\frac{\omega^*}{\tau}dt'. \tag{4.11}$$

Finally, as $t - t_1 > t_2$, we have $\theta(t,t_1) < \theta(t_2)$. Furthermore, by the definition of A, we have $\int_{t_1}^{t} \Phi(t,t')\omega^*/\tau dt' < A$. Therefore, by choosing ε, t_1 and t_2, it follows that, for all $t > t_1 + t_2$, we have

$$\begin{aligned} I(t) &< \frac{\varepsilon}{2} + Nt_1\theta(t_2)\frac{\omega_m}{\tau} + A(1+\varepsilon)(J+\varepsilon) \\ &< \frac{3\varepsilon}{2} + A(1+\varepsilon)(J+\varepsilon) < J - \frac{\varepsilon}{2}, \end{aligned} \tag{4.12}$$

contradiction. Therefore, $J = 0$ and $\lim_{t\to\infty} I(t) = 0$.

Remembering that, under the conditions of the statement, we have

$$S'(t) = \gamma N - \frac{\omega(t)S(t)I(t)}{N\tau} - \gamma S(t), \tag{4.13}$$

then $\lim_{t\to\infty} I(t) = 0$ implies that $\lim_{t\to\infty} S(t) = N$. Therefore, $\lim_{t\to\infty} R(t) = \lim_{t\to\infty}(N - I(t) - S(t)) = 0$. □

Theorem 3. *If $\omega(t) \equiv \omega$, $\gamma(t) \equiv \gamma$ and $\beta = 1$ in the system* (2.18)-(2.20)*, the endemic equilibrium of Eq.* (4.3) *is locally asymptotically stable whenever it is feasible, that is, when $\omega > (\tau\gamma)^{1-\alpha} + (\tau\gamma)$.*

Proof. We have $\Re_0 = \int_0^\infty \sigma(t)\Phi(t,0)dt = N/S^*$. We remember that if $\beta = 1$ and ω is constant, then $\sigma(t) \equiv \omega/\tau$. Thus,

$$\begin{aligned} I^* &= \int_0^\infty \sigma(t')\Phi(t')dt' \cdot \frac{S^*}{N} I^* = \int_0^\infty \Phi(t')\frac{S^*I^*\omega}{N\tau}dt' \\ &= \int_0^t \Phi(t-t')\frac{S^*I^*\omega}{N\tau}dt' + \int_t^\infty \Phi(t')\frac{S^*I^*\omega}{N\tau}dt'. \end{aligned} \tag{4.14}$$

Now, we observe that

$$\begin{aligned} \int_0^\infty \frac{\omega}{\tau}F(t')\theta(t',0)dt' &= \int_0^\infty \frac{\omega}{\tau}\theta(t',0)dt' - \frac{N}{S^*} = \frac{\omega}{\tau\gamma} - \frac{N}{S^*} \\ &= \frac{N}{S^*}(\tau\gamma)^{-\alpha} = (\tau\gamma)^{-\alpha}\int_0^\infty \frac{\omega}{\tau}\Phi(t')dt', \end{aligned} \tag{4.15}$$

i.e,

$$\frac{N}{S^*} = \int_0^\infty \frac{\omega}{\tau}\Phi(t')dt' = (\tau\gamma)^\alpha \int_0^\infty \frac{\omega}{\tau}F(t')\theta(t',0)dt'. \tag{4.16}$$

Returning to Eq. (4.3), we note that $R^* = (\tau\gamma)^{-\alpha}I^*$. So, we write

$$\begin{aligned} R^* &= (\tau\gamma)^{-\alpha}(\tau\gamma)^\alpha \int_0^\infty \frac{\omega}{\tau}F(t')\theta(t',0)dt' \cdot \frac{S^*}{N} \cdot I^* = \int_0^\infty F(t')\theta(t',0)\frac{S^*I^*\omega}{N\tau}dt' \\ &= \int_0^t F(t-t')\theta(t,t')\frac{S^*I^*\omega}{N\tau}dt' + \int_t^\infty F(t')\theta(t',0)\frac{S^*I^*\omega}{N\tau}dt'. \end{aligned} \tag{4.17}$$

We do $V = I - I^*$ and $W = R - R^*$ to translate the equilibrium to the origin. Then, by Eqs. (4.7), (4.8), (4.14) and (4.17) and observing that $SI - S^*I^* = S^*V - I^*(V+W) - V(V+W)$, we get

$$\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} + \int_0^t \begin{bmatrix} \Phi(t,t')\omega/N\tau & 0 \\ F(t-t')\theta(t,t')\omega/N\tau & 0 \end{bmatrix} \times \begin{bmatrix} (S^*V - I^*(V+W) - V(V+W))(t') \\ (V+W)(t') \end{bmatrix} dt', \tag{4.18}$$

on what

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} \Phi(t,0)i_0 - \int_t^\infty \Phi(t') \dfrac{S^*I^*\omega}{N\tau} dt' \\ F(t)\theta(t,0)i_0 - \int_t^\infty F(t')\theta(t',0) \dfrac{S^*I^*\omega}{N\tau} dt' \end{bmatrix}. \tag{4.19}$$

The nonlinear Volterra integral system of Eq. (4.18) can be written in matrix form as:

$$X(t) = F(t) + \int_0^t A(t-t')G(X(t'))dt', \tag{4.20}$$

on what

$$X = \begin{bmatrix} V \\ W \end{bmatrix}; F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}; A = \begin{bmatrix} \Phi\omega/N\tau & 0 \\ F\theta\omega/N\tau & 0 \end{bmatrix}; \tag{4.21}$$

$$G(X) = \begin{bmatrix} S^*V - I^*(V+W) - V(V+W) \\ V+W \end{bmatrix}. \tag{4.22}$$

The characteristic equation of the linearization of Eq. (4.20) is

$$\det\left(\text{Identity} - \int_0^\infty e^{-\lambda t} A(t) J dt\right) = 0, \tag{4.23}$$

where J is the Jacobian of G evaluated at 0.

The stability analysis of the origin uses a result cited in [10]: *If the solutions of Eq.* (4.20) *exist in* $[0,\infty)$ *and are bounded,* $F(t) \in C[0,\infty)$, $F(t) \to 0$ *when* $t \to \infty$, $A(t) \in L^1[0,\infty)$, $G(X) \in C^1(\mathbb{R}^2)$, $G(0) = 0$, *J is non-singular and Eq.* (4.23) *does not have roots with non-negative real parts, then the origin is a locally asymptotically stable equilibrium.*

Most of the conditions are straightforward. In particular, $F(t) \to 0$ due to the existence of the Laplace Transform of the Mittag-Leffler function. We need to prove that there are no characteristic roots of Eq. (4.23) with a non-negative real part. First, we calculate J, obtaining

$$J = \begin{bmatrix} S^* - I^* - 2V & -I^* - V \\ 1 & 1 \end{bmatrix}_{V=W=0} = \begin{bmatrix} S^* - I^* & -I^* \\ 1 & 1 \end{bmatrix}. \tag{4.24}$$

Thus, the characteristic equation is given by

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \int_0^\infty \begin{bmatrix} e^{-\lambda t}\Phi(t,0)\omega(S^* - I^*)/N\tau & -e^{-\lambda t}\Phi(t,0)I^*\omega/N\tau \\ e^{-\lambda t}F(t)e^{-\gamma t}\omega(S^* - I^*)/N\tau & -e^{-\lambda t}F(t)e^{-\gamma t}I^*\omega/N\tau \end{bmatrix} dt\right) = 0. \tag{4.25}$$

We observe that the integrals have the form of a Laplace transform and, in [10], is requested the hypothesis that the probability of permanence ϕ is dominated by an exponential decay, in order to guarantee the existence of the transform. The Mittag-Leffler function is not dominated by exponential decay, but the Laplace transform converges if $\mathrm{Re}(\lambda+\gamma)>0$, and $|\lambda+\gamma|>1/\tau$ [8].

Eq. (4.25) can be written as

$$\left(1-\int_0^\infty e^{-\lambda t}\Phi(t,0)(S^*-I^*)\frac{\omega}{N\tau}dt\right)\left(1+\int_0^\infty e^{-\lambda t}F(t)e^{-\gamma t}I^*\frac{\omega}{N\tau}dt\right)$$
$$-\left(\int_0^\infty e^{-\lambda t}\Phi(t,0)I^*\frac{\omega}{N\tau}dt\right)\left(-\int_0^\infty e^{-\lambda t}F(t)e^{-\gamma t}(S^*-I^*)\frac{\omega}{N\tau}dt\right)=0. \tag{4.26}$$

Which resumes to

$$1-\int_0^\infty e^{-\lambda t}\Phi(t,0)(S^*-I^*)\frac{\omega}{N\tau}dt+\int_0^\infty e^{-\lambda t}F(t)e^{-\gamma t}I^*\frac{\omega}{N\tau}dt=0. \tag{4.27}$$

Finally, since $\Phi(t,0)=\theta(t,0)\phi(t)=e^{-\gamma t}(1-F(t))$, we have

$$1-\int_0^\infty e^{-\lambda t}e^{-\gamma t}(S^*-I^*)\frac{\omega}{N\tau}dt+\int_0^\infty e^{-\lambda t}F(t)e^{-\gamma t}(S^*-I^*)\frac{\omega}{N\tau}dt$$
$$+\int_0^\infty e^{-\lambda t}F(t)e^{-\gamma t}I^*\frac{\omega}{N\tau}dt=0, \tag{4.28}$$

$$1-\int_0^\infty e^{-\lambda t}e^{-\gamma t}(1-F(t))S^*\frac{\omega}{N\tau}dt+\int_0^\infty e^{-\lambda t}e^{-\gamma t}I^*\frac{\omega}{N\tau}dt=0. \tag{4.29}$$

The condition $\omega>(\tau\gamma)^{1-\alpha}+(\tau\gamma)$ is equivalent to

$$\Re_0=\frac{N}{S^*}=\int_0^\infty\Phi(t,0)\frac{\omega}{\tau}dt=\int_0^\infty e^{-\gamma t}(1-F(t))\frac{\omega}{\tau}dt>1. \tag{4.30}$$

Supposing absurdly that $\mathrm{Re}(\lambda)\geq 0$, then

$$\int_0^\infty e^{-\gamma t}(1-F(t))\frac{\omega}{\tau}dt=\frac{N}{S^*}>\mathrm{Re}\left(\int_0^\infty e^{-\lambda t}e^{-\gamma t}(1-F(t))\frac{\omega}{\tau}dt\right). \tag{4.31}$$

Thus,

$$1>\mathrm{Re}\left(\int_0^\infty e^{-\lambda t}e^{-\gamma t}(1-F(t))S^*\frac{\omega}{N\tau}dt\right), \tag{4.32}$$

and, to fulfill Eq. (4.29), it will be necessary to have

$$\mathrm{Re}\left(\int_0^\infty e^{-\lambda t}e^{-\gamma t}I^*\frac{\omega}{N\tau}dt\right)<0. \tag{4.33}$$

But, see that, if $\mathrm{Re}(\lambda)>-\gamma$, we have

$$\int_0^\infty e^{-\lambda t}e^{-\gamma t}I^*\frac{\omega}{N\tau}dt=\frac{\omega I^*}{(\lambda+\gamma)N\tau}. \tag{4.34}$$

Since $\lambda = a + bi$ with $a \geq 0$, we have

$$\frac{\omega I^*}{(\lambda+\gamma)N\tau} = \frac{\omega I^*}{N\tau} \cdot \frac{1}{(a+\gamma)+bi} \cdot \frac{(a+\gamma)-bi}{(a+\gamma)-bi} = \frac{\omega I^*}{N\tau} \cdot \frac{(a+\gamma)-bi}{(a+\gamma)^2+b^2}, \tag{4.35}$$

following that

$$\mathrm{Re}\left(\frac{\omega I^*}{(\lambda+\gamma)N\tau}\right) = \frac{\omega I^*(a+\gamma)}{N\tau((a+\gamma)^2+b^2)} > 0, \tag{4.36}$$

contradiction with (4.33). Therefore, the equilibrium (S^*, I^*, R^*) is locally asymptotically stable.

□

5 DISCUSSION AND CONCLUSION

We discuss the SIR model from [3] with the aim of expanding on previous results and sharing the findings. This model is of interest to us because it is based on a meaningful construction that naturally involves derivatives of arbitrary order. We also demonstrate how the same fractional model can be derived from the theory presented in [15]. This approach helps to clarify the role of the proposed generalizations in the context of the classical SIR model.

In addition to introducing the model, we establish non-negativity and investigate the behavior of the compartments S and R, showing monotonicity in the limiting case where $\gamma \equiv 0$ is constant. This aligns with biological interpretations.

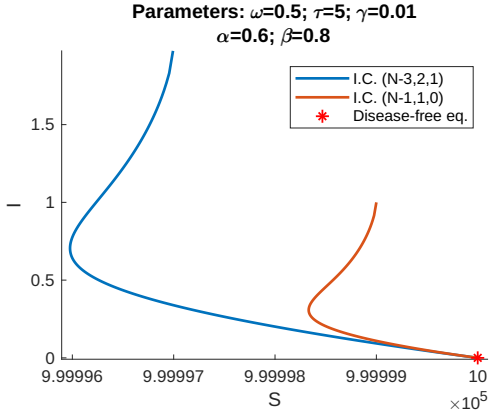

Figure 3: Trajectories $\Re_0 < 1$.

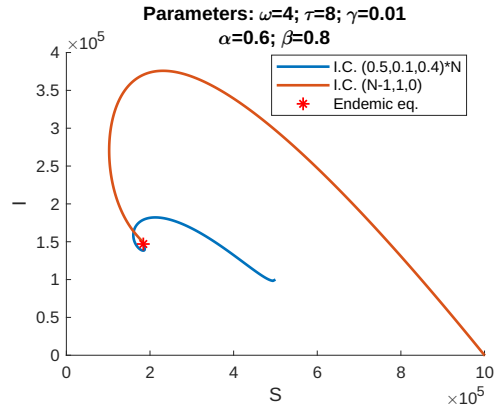

Figure 4: Trajectories $\Re_0 > 1$.

Equilibrium points are analyzed, and we prove global asymptotic stability for the disease-free equilibrium and local asymptotic stability for the endemic equilibrium, assuming simplifications such as $\gamma(t) \equiv \gamma$ (what is reasonable) and $\beta = 1$ (subject to further investigation due to the neglect of time-dependent infectivity). We hypothesize that the region with $i_0 > 0$ is a stable region for (S^*, I^*, R^*), even for $\beta < 1$. Figures 3 and 4 illustrate Theorems 2 and 3, with different initial conditions and $\beta < 1$.

Furthermore, a recent study by [24] provides a global analysis of the equilibria of an SIS model related to the model in equations (2.18)-(2.20) with $\beta = 1$, with the main difference being the existence of an entry depending on I in the S compartment. Their geometric approach to global stability problems may extend the results of Theorem 3 globally, although this extension is not straightforward. Also, [24] uses an approximation for the fractional derivative of I that needs to be further explored in the case of the SIR model. Other memory kernels, as those with $\alpha \leq 1 < \beta$, offer opportunities for generalizing the model to various types of diseases.

Finally, we emphasize that the model in (2.40) uses the tempered fractional Riemann-Liouville derivative with an exponential tempering function. This is a notable application of Tempered Fractional Calculus, an extension of classical Fractional Calculus [21]. Additionally, Eq. (2.12) relates to an $\alpha-$exponential function, the usefulness of which has been recently discussed [4].

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