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Calculation of Green's Function for Poisson's Equation in Plane Polar Coordinates

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ABSTRACT. A new calculation of Green's function for Poisson's equation in plane polar coordinates is presented. The method consists in first calculating the solution to the simpler problem, but with the same Green's function, that is obtained with the homogenization of the boundary conditions and then inferring Green's function by comparing this calculated solution with Green's solution formula. Depending on how the solution to the simplified problem is calculated, Green's function may result as an integral or an infinite series, but it is finally presented in a closed form, because it is possible to calculate the integral or the sum of this series.

Keywords: Green's function, Poisson, plane polar coordinates, disc sector, closed form, Dirichlet, Neumann.

1 INTRODUCTION

This work describes a new calculation of Green's function for Poisson's equation in plane polar coordinates in which it is obtained in closed form. To explain the method, we consider the domain Ω of the problem to be the disc sector shown in Figure 1 as well as the boundary conditions to be those indicated there: Dirichlet's on the rectilinear boundary along the x-axis and on the circular boundary, and Neumann's on the other rectilinear boundary. This problem is formulated as follows:

$$\nabla^2 u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = h(r,\theta),\ r \in (0,b),\ \theta \in (0,\gamma)\ ; \tag{1.1a}$$

$$u(r,0) = f_0(r),\ r \in [0,b)\ ; \tag{1.1b}$$

$$\frac{\partial u}{\partial \theta}(r,\gamma) = g_\gamma(r),\ r \in (0,b)\ ; \tag{1.1c}$$

$$u(b,\theta) = f_b(\theta),\ \theta \in [0,\gamma]\ ; \tag{1.1d}$$

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where the functions f_0, f_b, g_γ, and h are continuous. We want a solution u that is continuous in $\Omega \cup \partial\Omega$. In this work, we use the following notation: by writing (1.1), we refer to equations (1.1a)–(1.1d) and therefore to the problem defined by them.

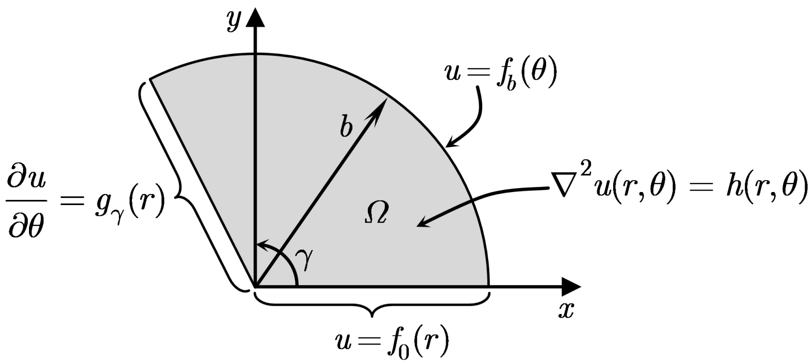

Figure 1: The problem for which Green's function is calculated in this work: that defined in equations (1.1a)–(1.1d).

Green's formula for the solution $u(\mathbf{r}) = u(r,\theta)$ is as follows:

$$2\pi u(\mathbf{r}) = -\int_\Omega G(\mathbf{r}|\mathbf{r}')h(\mathbf{r}')dA' - \int_{\partial\Omega_D} \frac{\partial G}{\partial n'}(\mathbf{r}|\mathbf{r}')f(\mathbf{r}')ds' + \int_{\partial\Omega_N} G(\mathbf{r}|\mathbf{r}')g(\mathbf{r}')ds', \qquad (1.2)$$

where $\partial\Omega_D$ is the part of $\partial\Omega$ under Dirichlet's conditions (on which $f = f_0$ when $\theta = 0$ and $f = f_b$ when $r = b$), $\partial\Omega_N$ is the part under Neumann's conditions (on which $g = g_\gamma$), and $G(\mathbf{r}|\mathbf{r}')$ is defined as the solution to the problem

$$\nabla'^2 G(\mathbf{r}|\mathbf{r}') = -2\pi\delta(\mathbf{r}' - \mathbf{r}),\ G(\mathbf{r}|\mathbf{r}') = 0 \text{ for } \mathbf{r}' \in \partial\Omega_D,\ \frac{\partial G}{\partial n'}(\mathbf{r}|\mathbf{r}') = 0 \text{ for } \mathbf{r}' \in \partial\Omega_N. \qquad (1.3)$$

This formulation easily follows from *Green's representation formula* (which is a well-established result in the literature; see, for example, eq. (1.42) in Ref. [10].) Notice, however, that, in order to get the above eq. (1.2) from that eq. (1.42) in Ref. [10], we need to multiply this equation by $4\pi/2\pi$ and replace $-\rho/\varepsilon_0$ with h, because our problem (1.1) is two-dimensional (and not tridimensional) and exhibits simply h (instead of $-\rho/\varepsilon_0$) in the right-hand side of Poisson's equation. A rigorous development of Green's function method for Poisson's equation is presented in Section 2.2.4 of Ref. [7].

The method developed in this work takes advantage of the fact that Green's function $G(\mathbf{r}|\mathbf{r}')$ for problem (1.1) does not depend on the boundary data formed by the functions f_0, f_b, g_γ, and h. This statement is a direct consequence of the fact that the definition of Green's function in (1.3) does not involve those functions. Then, to calculate $G(\mathbf{r}|\mathbf{r}')$, we consider the following simplified version of problem (1.1), in which all boundary conditions are homogeneous:

$$\nabla^2 v(r,\theta) = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = h(r,\theta),\ r\in(0,b),\ \theta\in(0,\gamma)\ ; \tag{1.4a}$$

$$v(r,0) = 0,\ r\in[0,b)\ ; \tag{1.4b}$$

$$\frac{\partial v}{\partial \theta}(r,\gamma) = 0,\ r\in(0,b)\ ; \tag{1.4c}$$

$$v(b,\theta) = 0,\ \theta\in[0,\gamma]\ . \tag{1.4d}$$

The first step of the method is the calculation of the solution $v(r,\theta)$ to the above problem. In this work, we solve this problem using *eigenfunction expansion*, and we do it twice, first using angular eigenfunctions and then radial eigenfunctions.

The second step is the determination of the desired Green's function $G(\mathbf{r}|\mathbf{r}') = G(r,\theta\,|\,r',\theta')$ [which is the same for both problems (1.1) and (1.4)] from the solution $v(\mathbf{r})$ to problem (1.4) calculated in the previous step. This is done as follows: Since, according to (1.2), Green's formula for the solution $v(r,\theta)$ to problem (1.4) is simply given by

$$\begin{aligned} v(r,\theta) &= -\frac{1}{2\pi}\iint_{\Omega} G(\mathbf{r}|\mathbf{r}')\,h(\mathbf{r}')\,dA' \\ &= -\frac{1}{2\pi}\int_0^{\gamma} d\theta' \int_0^b dr'\,r'\,h(r',\theta')\,G(r,\theta\,|\,r',\theta')\ , \end{aligned} \tag{1.5}$$

it is possible to infer an expression for $G(r,\theta|r,\theta')$ by writing the already calculated solution $v(r,\theta)$ in the form of the double integral on the right side of the above equation. We will see that this writing is not an automatic task, requiring some artifices in the first step.

A third step is still necessary, because the Green's function expression obtained in the second step may involve an integral or an infinite series. We therefore need to evaluate this integral or the sum of this series to obtain Green's function in closed form.

In what follows, in Sections 2 and 3, we apply the method outlined above to calculate Green's function for problem (1.1) expanded into angular and radial eigenfunctions, respectively. Section 4 shows how to compute the sum of the infinite series in the Green's function expression calculated in the previous two sections to finally present it in closed form. Section 5 contains a comparison of this closed-form Green's function with the one provided by the method of images for the particular case when $\gamma = \pi/2$. Section 6 briefly exposes the application of the method to an extra problem. Section 7 ends the body of the paper with final conclusions.

2 THE CALCULATION BASED ON THE ANGULAR EIGENFUNCTIONS

In this section, we perform the first step of the method: the calculation of the solution to problem (1.4). To this end, we admit that this solution can be expanded into the eigenfunctions $\Theta_n(\theta) = \sin(n\pi\theta/2\gamma)$ $(n = 1,3,5\cdots)$ {cf. Ref. [2], Sec. 10.1, Prob. 19, p. 595 & 786} that arise when the separation of variables $v(r,\theta) = R(r)\Theta(\theta)$ is used to solve the version of problem (1.4) in which $h(r,\theta) \equiv 0$ (Laplace's equation) and the boundary condition on the circular boundary (at $r = b$) is not homogeneous. That is, we admit that

$$v(r,\theta) = \sum_{n=1,3,5\cdots} v_n(r) \sin\frac{n\pi\theta}{2\gamma} \ . \tag{2.1}$$

Notice that this expression automatically satisfies the conditions of problem (1.4) on the boundaries at $\theta = 0$ and $\theta = \gamma$.

By substituting (2.1) into the partial differential equation of the problem, we get

$$\sum_{n=1,3,5\cdots} \left[v_n'' + \frac{1}{r} v_n' - \frac{(n\pi/2\gamma)^2}{r^2} v_n \right] \sin\frac{n\pi\theta}{2\gamma} = h \ .$$

This result shows that the terms in brackets for $n = 1,3,5\cdots$ are the coefficients of a generalized Fourier sine series of the function h; therefore,

$$v_n'' + \frac{1}{r} v_n' - \frac{(n\pi/2\gamma)^2}{r^2} v_n(r) \ = \ \frac{2}{\gamma} \int_0^\gamma h(r,\theta) \sin\frac{n\pi\theta}{2\gamma}\, d\theta \ \equiv \ h_n(r) \ . \tag{2.2}$$

We thus see that $v_n(r)$ is the solution to a nonhomogeneous Euler-Cauchy differential equation [9, Sec. 1.6].

Since the general solution to the associated homogeneous equation is

$$v_{Hn}(r) = c_{1n} r^{n\pi/2\gamma} + c_{2n}/r^{n\pi/2\gamma} \ ,$$

a particular solution by the method of variation of parameters [9, Sec. 1.9] has the form

$$v_{Pn}(r) = A_n(r)\, r^{n\pi/2\gamma} + B_n(r)/r^{n\pi/2\gamma} \ , \tag{2.3}$$

where the functions $A_n(r)$ and $B_n(r)$ are solutions to the system of equations

$$\begin{cases} A_n' r^{n\pi/2\gamma} + B_n'/r^{n\pi/2\gamma} = 0 \\ (n\pi/2\gamma) A_n' r^{(n\pi/2\gamma)-1} - (n\pi/2\gamma) B_n'/r^{(n\pi/2\gamma)+1} = h_n \ . \end{cases}$$

Solving it, we get

$$A_n'(r) = \frac{\gamma h_n(r)}{n\pi r^{(n\pi/2\gamma)-1}} \ \Rightarrow \ A_n(r) = \frac{\gamma}{n\pi} \int_0^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}} dr' , \tag{2.4a}$$

$$B_n'(r) = -\frac{\gamma h_n(r)\, r^{(n\pi/2\gamma)+1}}{n\pi} \ \Rightarrow \ B_n(r) = -\frac{\gamma}{n\pi} \int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1} dr' \ . \tag{2.4b}$$

Using these results in (2.3), we can write the general solution $v_{Hn}(r) + v_{Pn}(r)$ to (2.2) as

$$v_n(r) = \left[c_{1n} + \frac{\gamma}{n\pi} \int_0^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}} dr' \right] r^{n\pi/2\gamma} + \left[c_{2n} - \frac{\gamma}{n\pi} \int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1} dr' \right] \frac{1}{r^{n\pi/2\gamma}} \ . \tag{2.5}$$

To determine c_{1n} and c_{2n}, we impose the conditions of the problem, first the one related to continuity. To prevent (2.5) from becoming infinite when $r \to 0$, it is necessary that

$$\lim_{r\to 0}\left[c_{2n} - \frac{\gamma}{n\pi}\int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1} dr'\right] = 0 \quad \Rightarrow \quad c_{2n} = 0 \ .$$

With this result, (2.5) becomes

$$v_n(r) = \left[c_{1n} + \frac{\gamma}{n\pi}\int_0^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}} dr'\right] r^{n\pi/2\gamma} \\ + \left[-\frac{\gamma}{n\pi}\int_0^r h_n(r')\, r'^{(n\pi/2\gamma)+1} dr'\right]\frac{1}{r^{n\pi/2\gamma}} \ . \tag{2.6}$$

Now we require that (2.6) satisfies the condition

$$v_n(b) = 0 \ , \tag{2.7}$$

which results from the substitution of (2.1) into the condition $v(b,\theta) = 0$ given by (1.4d). We obtain

$$c_{1n} = \frac{\gamma}{n\pi}\int_0^b dr' r' h_n(r')\left[\left(\frac{r'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{1}{r'}\right)^{\frac{n\pi}{2\gamma}}\right] \ .$$

Using this expression for c_{1n} in (2.6), we can write $v_n(r)$ as follows:

$$v_n(r) = \frac{\gamma}{n\pi}\int_0^b dr' r' h_n(r')\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r}{r'}\right)^{\frac{n\pi}{2\gamma}}\right] \\ + \frac{\gamma}{n\pi}\int_0^r dr' r' h_n(r')\left[\left(\frac{r}{r'}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r'}{r}\right)^{\frac{n\pi}{2\gamma}}\right] \ . \tag{2.8}$$

This expression for $v_n(r)$ is not suitable to express $v(r,\theta)$ in the form of the double integral in (1.5), because, in that double integral, the interval of integration with respect to r' is $[0,b]$, whereas, in the second integral above, it is $[0,r]$. To overcome this difficulty we derive another expression for $v_n(r)$ with a slightly different form as follows: Since the lower limit of integration of the indefinite integrals in (2.4) is an arbitrary point of $[0,b]$, we choose it now to be b (instead of 0) to obtain, instead of (2.5), this other equivalent expression for the general solution to (2.2):

$$v_n(r) = \left[d_{1n} + \frac{\gamma}{n\pi}\int_b^r \frac{h_n(r')}{r'^{(n\pi/2\gamma)-1}} dr'\right] r^{n\pi/2\gamma} \\ + \left[d_{2n} - \frac{\gamma}{n\pi}\int_b^r h_n(r')\, r'^{(n\pi/2\gamma)+1} dr'\right]\frac{1}{r^{n\pi/2\gamma}} \ . \tag{2.9}$$

As before, to prevent this expression from becoming infinite as $r \to 0$, it is necessary that

$$\lim_{r\to 0}\left[d_{2n} - \frac{\gamma}{n\pi}\int_b^r h_n(r')\, r'^{(n\pi/2\gamma)+1} dr'\right] = 0 \ \Rightarrow \ d_{2n} = \frac{\gamma}{n\pi}\int_b^0 h_n(r')\, r'^{(n\pi/2\gamma)+1} dr',$$

and, by imposing condition (2.7) on (2.9) and then substituting the above expression for d_{2n}, we get

$$v_n(b) = d_{1n} b^{n\pi/2\gamma} + \frac{d_{2n}}{b^{n\pi/2\gamma}} = 0 \;\Rightarrow\; d_{1n} = -\frac{\gamma}{n\pi (b^2)^{n\pi/2\gamma}} \int_b^0 h_n(r')\, r'^{(n\pi/\gamma)+1} dr' \, .$$

Using these results for d_{1n} and d_{2n} in (2.9), we can write the following expression for $v_n(r)$:

$$v_n(r) = \frac{\gamma}{n\pi} \int_0^b dr' r' h_n(r') \left[\left(\frac{rr'}{b^2} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r'}{r} \right)^{\frac{n\pi}{2\gamma}} \right] + \frac{\gamma}{n\pi} \int_r^b dr' r' h_n(r') \left[\left(\frac{r'}{r} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r}{r'} \right)^{\frac{n\pi}{2\gamma}} \right] . \quad (2.10)$$

Now we have both (2.8) and (2.10) expressing $v_n(r)$. The idea is to add these two equations and then replace the sum of the integrals $\int_0^r$ and $\int_r^b$ with a single integral of the form $\int_0^b$, whose interval of integration is the one in (1.5). Note, however, that this cannot yet be done, because the integrands of these two integrals are not exactly the same. But since one becomes the other by replacing r/r' with r'/r, and since $r \geq r'$ in the first integral and $r \leq r'$ in the second integral, one way to make these two integrals display the same integrand is to define

$$r_< \; (r_>) \equiv \text{the smaller (larger) of } r \text{ and } r' \; . \quad (2.11)$$

In fact, with this notation, because $r = r_>$ and $r' = r_<$ in the integral $\int_0^r dr'$, and $r = r_<$ and $r' = r_>$ in $\int_r^b dr'$, we have that (2.8) and (2.10) are respectively given by

$$v_n(r) = \frac{\gamma}{n\pi} \int_0^b dr' r' h_n(r') \left[\left(\frac{rr'}{b^2} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r}{r'} \right)^{\frac{n\pi}{2\gamma}} \right] + \frac{\gamma}{n\pi} \int_0^r dr' r' h_n(r') \left[\left(\frac{r_>}{r_<} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>} \right)^{\frac{n\pi}{2\gamma}} \right]$$

and

$$v_n(r) = \frac{\gamma}{n\pi} \int_0^b dr' r' h_n(r') \left[\left(\frac{rr'}{b^2} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r'}{r} \right)^{\frac{n\pi}{2\gamma}} \right] + \frac{\gamma}{n\pi} \int_r^b dr' r' h_n(r') \left[\left(\frac{r_>}{r_<} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>} \right)^{\frac{n\pi}{2\gamma}} \right] .$$

Therefore, adding these two equations, and again using (2.11), we get the proper form of the expression of $v_n(r)$ to be used in (1.5):

$$\begin{aligned} 2\, v_n(r) \;&=\; \frac{\gamma}{n\pi} \int_0^b dr' r' h_n(r') \left[2 \left(\frac{rr'}{b^2} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>} \right)^{\frac{n\pi}{2\gamma}} - \cancel{\left(\frac{r_>}{r_<} \right)^{\frac{n\pi}{2\gamma}}} \right] \\ &\quad + \frac{\gamma}{n\pi} \int_0^b dr' r' h_n(r') \left[\cancel{\left(\frac{r_>}{r_<} \right)^{\frac{n\pi}{2\gamma}}} - \left(\frac{r_<}{r_>} \right)^{\frac{n\pi}{2\gamma}} \right] \\ \Rightarrow \quad v_n(r) \;&=\; \frac{\gamma}{n\pi} \int_0^b dr' r' h_n(r') \left[\left(\frac{rr'}{b^2} \right)^{\frac{n\pi}{2\gamma}} - \left(\frac{r_<}{r_>} \right)^{\frac{n\pi}{2\gamma}} \right] . \end{aligned} \quad (2.12)$$

Taking this result into (2.1) and using the defining expression for h_n given in (2.2), we get

$$
\begin{aligned}
v(r,\theta) \;=\;& \\
&\sum_{n=1,3,5\cdots}\left\{\frac{\gamma}{n\pi}\int_0^b dr'r'\left(\frac{2}{\gamma}\int_0^\gamma d\theta' h(r',\theta')\sin\frac{n\pi\theta'}{2\gamma}\right)\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}}-\left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}\right]\right\}\sin\frac{n\pi\theta}{2\gamma} \\
=& -\frac{1}{2\pi}\int_0^\gamma d\theta'\int_0^b dr'r'h(r',\theta')\underbrace{\sum_{n=1,3,5\cdots}\frac{4}{n}\left[\left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}-\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}}\right]\sin\frac{n\pi\theta'}{2\gamma}\sin\frac{n\pi\theta}{2\gamma}}_{G(r,\theta\,|\,r',\theta')}\,,
\end{aligned}
\tag{2.13}
$$

from which, by comparison with (1.5), we infer the expression for $G(r,\theta\,|\,r',\theta')$ indicated above:

$$
G(r,\theta\,|\,r',\theta')=\sum_{n=1,3,5\cdots}\frac{4}{n}\left[\left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}}-\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}}\right]\sin\frac{n\pi\theta'}{2\gamma}\sin\frac{n\pi\theta}{2\gamma}\ . \tag{2.14}
$$

3 THE CALCULATION BASED ON THE RADIAL EIGENFUNCTIONS

In this section we again calculate the solution to (1.4), now in the form of an expansion in radial eigenfunctions. But, instead of using these eigenfunctions as functions of r, we find it more convenient to consider them as functions of the variable ρ that is related to r as follows:

$$
\rho = -\ln(r/b) \in [0,\infty) \quad\Leftrightarrow\quad r = be^{-\rho} \in [0,b]\ . \tag{3.1}
$$

Using the notation

$$
v(r,\theta) = v(be^{-\rho},\theta) \equiv V(\rho,\theta)
$$

and applying the chain rule, we obtain

$$
r\frac{\partial v}{\partial r}(r,\theta) = -\frac{\partial V}{\partial \rho}(\rho,\theta) \quad\text{and}\quad r^2\frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 V}{\partial \rho^2}+\frac{\partial V}{\partial \rho}\ .
$$

Consequently, Poisson's equation in (1.4) takes the simpler form

$$
r^2\nabla^2 v(r,\theta) = \frac{\partial^2 V}{\partial \rho^2}+\frac{\partial^2 V}{\partial \theta^2}(\rho,\theta) = r^2 h(r,\theta) \equiv H(\rho,\theta)
$$

(with constant coefficients), and the homogeneous boundary conditions are given by

$$
V(\rho,0) = v(r,0) = 0\,,\ \ \frac{\partial V}{\partial \theta}(\rho,\gamma) = \frac{\partial v}{\partial \theta}(r,\gamma) = 0\,,\ \text{and}\ \ V(0,\theta) = v(b,\theta) = 0\,.
$$

Therefore, problem (1.4) in terms of the variable ρ becomes

$$
\frac{\partial^2 V}{\partial \rho^2}+\frac{\partial^2 V}{\partial \theta^2}(\rho,\theta) = H(\rho,\theta),\ \rho\in(0,\infty),\ \theta\in(0,\gamma)\,; \tag{3.2a}
$$

$$
V(\rho,0) = \frac{\partial V}{\partial \theta}(\rho,\gamma) = 0,\ \rho\in(0,\infty)\,; \tag{3.2b}
$$

$$
V(0,\theta) = 0,\ \theta\in[0,\gamma]\,. \tag{3.2c}
$$

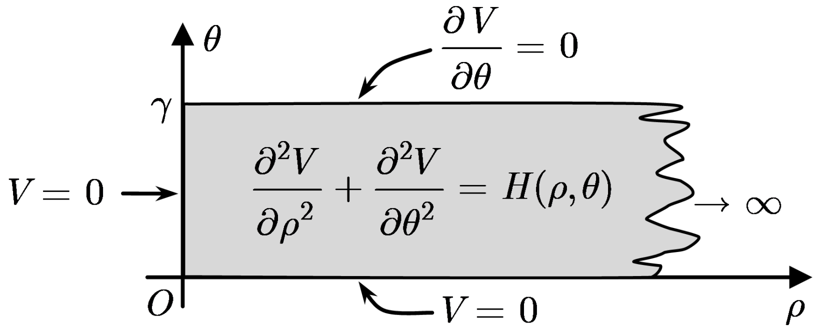

Figure 2: Depiction of the problem in (3.2).

This problem is depicted in Figure 2.

Green's formula for the solution $V(\rho,\theta)$ to problem (3.2) follows from (1.5) by making the change from variables r and r' to ρ and ρ' [we omit the variables of Green's function, simply denoting it by G, regardless of its dependence on $(r,\theta\,|\,r',\theta')$ or $(\rho,\theta\,|\,\rho',\theta')$],

$$V(\rho,\theta) = -\frac{1}{2\pi}\int_0^{\gamma} d\theta' \int_{\infty}^{0} [-d\rho'\, r']\, \underbrace{r' h(r',\theta')}_{H(\rho',\theta')}\, G$$

$$= -\frac{1}{2\pi}\int_0^{\gamma} d\theta' \int_0^{\infty} d\rho'\, H(\rho',\theta')\, G \ . \tag{3.3}$$

To solve (3.2) we employ the eigenfunctions that arise when the separation of variables $V(\rho,\theta) = R(\rho)\Theta(\theta)$ is used to solve the version of this problem in which $H(\rho,\theta) \equiv 0$ (Laplace's equation) and the boundary conditions on the parallel borders in Figure 2 are not homogeneous. That is, we admit that the solution to problem (3.2) can be expanded into the continuous-spectrum eigenfunctions $R_k(\rho) = \sin k\rho \ (k > 0)$ {cf. Ref. [5], Sec. 8.7}:

$$V(\rho,\theta) = \int_0^{\infty} \bar{V}(k,\theta)\sin k\rho\, dk \ . \tag{3.4}$$

Notice that this expression automatically satisfies the condition of problem (3.2) on the boundary at $\rho = 0$.

By substituting (3.4) into the PDE of problem (3.2), we get

$$\int_0^{\infty} \left[\frac{\partial^2 \bar{V}}{\partial \theta^2} - k^2 \bar{V}(k,\theta)\right] \sin k\rho\, dk = H(\rho,\theta) \ .$$

This result shows that the term in square brackets, according to the Fourier sine integral representation of $H(\rho,\theta)$, is given by {cf. Ref. [4], Sec. 54, eqs. (5) and (6)}

$$\frac{\partial^2 \bar{V}}{\partial \theta^2} - k^2 \bar{V}(k,\theta) = \frac{2}{\pi}\int_0^{\infty} H(\rho,\theta)\sin k\rho\, d\rho \equiv \bar{H}(k,\theta) \ . \tag{3.5}$$

We thus see that $\bar{V}(k,\theta)$ is the solution to the above nonhomogeneous ordinary differential equation.

Since the general solution to the associated homogeneous differential equation is

$$\bar{V}_H(k,\theta) = c_1 \cosh k\theta + c_2 \sinh k\theta \ ,$$

a particular solution by the method of variation of parameters {cf. Ref. [9], Sec. 1.9} has the form

$$\bar{V}_P(k,\theta) = A(\theta)\cosh k\theta + B(\theta)\sinh k\theta \ , \tag{3.6}$$

where the functions $A(\theta)$ and $B(\theta)$ are solutions to the system of equations

$$\begin{cases} A'\cosh k\theta + B'\sinh k\theta = 0 \\ kA'\sinh k\theta + kB'\cos k\theta = \bar{H} \end{cases}$$

Solving it, we get

$$A'(\theta) = -\frac{\bar{H}\sin k\theta}{k} \quad \Rightarrow \quad A(\theta) = -\frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k\theta'\, d\theta' \ , \tag{3.7a}$$

$$B'(\theta) = \frac{\bar{H}\cosh k\theta}{k} \quad \Rightarrow \quad B(\theta) = \frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\cosh k\theta'\, d\theta' \ . \tag{3.7b}$$

With the substitution of these results into (3.6), we obtain

$$\begin{aligned}\bar{V}_P(k,\theta) &= -\frac{\cosh k\theta}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k\theta' d\theta' + \frac{\sinh k\theta}{k}\int_0^\theta \bar{H}(k,\theta')\cosh k\theta' d\theta' \\ &= -\frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k(\theta'-\theta)\, d\theta' \ .\end{aligned}$$

The general solution $\bar{V}_H(k,\theta) + \bar{V}_P(k,\theta)$ to (3.5) is then given by

$$\bar{V}(k,\theta) = c_1\cosh k\theta + c_2\sinh k\theta - \frac{1}{k}\int_0^\theta \bar{H}(k,\theta')\sinh k(\theta'-\theta)\, d\theta' \ . \tag{3.8}$$

To determine c_1 and c_2, we impose the homogeneous boundary conditions at $\theta = 0$ and $\theta = \gamma$ listed in (3.2) [those on the two parallel borders in Figure 2]. To this end, we substitute them into (3.4), obtaining

$$\bar{V}(k,0) = \frac{\partial \bar{V}}{\partial\theta}(k,\gamma) = 0 \ . \tag{3.9}$$

Imposing these two conditions on the expression of $\bar{V}(k,\theta)$ given by (3.8), we get the following expressions for c_1 and c_2:

$$\bar{V}(k,0) = c_1 = 0$$

$$\Rightarrow \quad \frac{\partial\bar{V}}{\partial\theta}(k,\gamma) = kc_2\cosh k\gamma + \int_0^\gamma \bar{H}(k,\theta')\cosh k(\theta'-\gamma)\, d\theta' = 0$$

$$\Rightarrow \quad c_2 = -\frac{1}{k\cosh k\gamma}\int_0^\gamma \bar{H}(k,\theta')\cosh k(\theta'-\gamma)\, d\theta' \ .$$

Substituting these results into (3.8), we obtain

$$\bar{V}(k,\theta) = -\frac{\sinh k\theta}{k\cosh k\gamma}\int_0^\gamma d\theta'\bar{H}(k,\theta')\cosh k(\theta'-\gamma) + \frac{1}{k}\int_0^\theta d\theta'\bar{H}(k,\theta')\sinh k(\theta-\theta')\,. \tag{3.10}$$

This expression for $\bar{V}(k,\theta)$ is not suitable to express $V(\rho,\theta)$ in the form of the double integral in (3.3), because, in that double integral, the interval of integration with respect to θ' is $[0,\gamma]$, whereas, in the second integral above, it is $[0,\theta]$. To overcome this difficulty we derive another expression for $\bar{V}(k,\theta)$ with a slightly different form as follows: Since the lower limit of integration of the indefinite integrals in (3.7) is an arbitrary point of $[0,\gamma]$, we choose it now to be γ (instead of 0) to obtain, instead of (3.8), the following equivalent expression for the general solution to (3.5):

$$\bar{V}(k,\theta) = d_1\cosh k\theta + d_2\sinh k\theta - \frac{1}{k}\int_\gamma^\theta \bar{H}(k,\theta')\sinh k(\theta'-\theta)\,d\theta'\,. \tag{3.11}$$

As before, by imposing the conditions given in (3.9), we determine d_1 and d_2:

$$\bar{V}(k,0) = d_1 - \frac{1}{k}\int_\gamma^0 \bar{H}(k,\theta')\sinh k\theta'\,d\theta' \;\Rightarrow\; d_1 = -\frac{1}{k}\int_0^\gamma \bar{H}(k,\theta')\sinh k\theta'\,d\theta'\,;$$

$$\frac{\partial\bar{V}}{\partial\theta}(k,\gamma) = kd_1\sinh k\gamma + kd_2\cosh k\gamma = 0 \quad\Rightarrow\quad d_2 = -\frac{\sinh k\gamma}{\cosh k\gamma}d_1$$

$$\Rightarrow\; d_2 = \frac{\sinh k\gamma}{k\cosh k\gamma}\int_0^\gamma \bar{H}(k,\theta')\sinh k\theta'\,d\theta'\,.$$

The substitution of these two results into (3.11) gives

$$\bar{V}(k,\theta) = -\frac{\cosh k(\gamma-\theta)}{k\cosh k\gamma}\int_0^\gamma d\theta'\bar{H}(k,\theta')\sinh k\theta' + \frac{1}{k}\int_\theta^\gamma d\theta'\bar{H}(k,\theta')\sinh k(\theta'-\theta)\,. \tag{3.12}$$

Now we have both (3.10) and (3.12) expressing $\bar{V}(k,\theta)$. The idea is to add these two equations and then replace the sum of the integrals $\int_0^\theta$ and $\int_\gamma^\theta$ with a single integral $\int_0^\gamma$, whose interval of integration is the one in (3.2). Note, however, that this cannot yet be done, because the integrands of these two integrals are not exactly the same. But since one becomes the other by replacing $\theta'-\theta$ with $\theta-\theta'$, and since $\theta \geq \theta'$ in the first integral and $\theta \leq \theta'$ in the second integral, one way to make these two integrals have the same integrand is by defining

$$\theta_< \; (\theta_>) \equiv \text{the smaller (larger) of } \theta \text{ and } \theta'\,.$$

In fact, with this notation, because $\theta = \theta_>$ and $\theta' = \theta_<$ in the integral $\int_0^\theta d\theta'$, and $\theta = \theta_<$ and $\theta' = \theta_>$ in $\int_\theta^\gamma d\theta'$, (3.10) and (3.12) are respectively given by

$$\bar{V}(k,\theta) = -\frac{1}{k\cosh k\gamma}\int_0^\gamma d\theta' \bar{H}(k,\theta')\sinh k\theta \cosh k(\theta'-\gamma) \\ + \frac{1}{k}\int_0^\theta d\theta' \bar{H}(k,\theta')\sinh k(\theta_> - \theta_<)$$

and

$$\bar{V}(k,\theta) = -\frac{1}{k\cosh k\gamma}\int_0^\gamma d\theta' \bar{H}(k,\theta')\cosh k(\gamma-\theta)\sinh k\theta' \\ + \frac{1}{k}\int_\theta^\gamma d\theta' \bar{H}(k,\theta')\sinh k(\theta_> - \theta_<) \ .$$

Therefore, adding these two equations we get the appropriate form of the expression for $\bar{V}(k,\theta)$ to be used in (3.4):

$$2\bar{V}(k,\theta) = -\frac{1}{k\cosh k\gamma}\int_0^\gamma d\theta' \bar{H}(k,\theta')[\sinh k\theta \cosh k(\theta'-\gamma) + \cosh k(\gamma-\theta)\sinh k\theta'] \\ + \frac{1}{k}\int_0^\gamma d\theta' \bar{H}(k,\theta')\sinh k(\theta_> - \theta_<) \ ,$$

or

$$\bar{V}(k,\theta) = \int_0^\gamma d\theta' \bar{H}(k,\theta')\frac{\Gamma(k,\theta,\theta')}{2k\cosh k\gamma} \ , \tag{3.13}$$

where

$$\Gamma(k,\theta,\theta') \equiv -\sinh k\theta \cosh k(\theta'-\gamma) - \cosh k(\gamma-\theta)\sinh k\theta' \\ + \cosh k\gamma \sinh k(\theta_> - \theta_<) \ . \tag{3.14}$$

Taking (3.13) into (3.4) and then using the definition of $\bar{H}(k,\theta')$ given by (3.5), we obtain

$$\begin{aligned} V(\rho,\theta) &= \int_0^\infty \left\{\bar{V}(k,\theta)\right\}\sin k\rho \, dk = \int_0^\infty \left\{\int_0^\gamma d\theta' \left[\bar{H}(k,\theta')\right]\frac{\Gamma(k,\theta,\theta')}{2k\cosh k\gamma}\right\}\sin k\rho \, dk \\ &= \int_0^\infty \left\{\int_0^\gamma d\theta' \left[\frac{2}{\pi}\int_0^\infty H(\rho',\theta')\sin k\rho' \, d\rho'\right]\frac{\Gamma(k,\theta,\theta')}{2k\cosh k\gamma}\right\}\sin k\rho \, dk \\ &= -\frac{1}{2\pi}\int_0^\gamma d\theta' \int_0^\infty d\rho' H(\rho',\theta') \underbrace{\int_0^\infty dk \frac{-2\Gamma(k,\theta,\theta')\sin k\rho' \sin k\rho}{k\cosh k\gamma}}_{G} \ . \end{aligned}$$

By comparing this result with (3.2), we infer that Green's function is as indicated above by G, but, before displaying it, let us write (3.14) in terms of $\theta_<$ and $\theta_>$ only, that is, without the explicit presence of θ or θ':

$$\begin{aligned}
\Gamma(k,\theta,\theta') &= -\sinh k\theta \cosh k(\theta'-\gamma) - \cosh k(\gamma-\theta)\sinh k\theta' \\
&\quad + \cosh k\gamma \sinh k(\theta_> - \theta_<) \\
&= -\sinh k\theta \cosh k\theta' \cosh k\gamma + \sinh k\theta \sinh k\theta' \sinh k\gamma - \cosh k\theta \cosh k\gamma \sinh k\theta' \\
&\quad + \sinh k\theta \sinh k\gamma \sinh k\theta' + \cosh k\gamma \sinh k(\theta_> - \theta_<) \\
&= 2\sinh k\theta_> \sinh k\theta_< \sinh k\gamma - \cosh k\gamma(\cancel{\sinh k\theta_> \cosh k\theta_<} + \sinh k\theta_< \cosh k\theta_>) \\
&\quad + \cosh k\gamma(\cancel{\sinh k\theta_> \cosh k\theta_<} - \sinh k\theta_< \cosh k\theta_>) \\
&= 2\sinh k\theta_> \sinh k\theta_< \sinh k\gamma - 2\cosh k\gamma \sinh k\theta_< \cosh k\theta_> \\
&= 2\sinh k\theta_< (\sinh k\theta_> \sinh k\gamma - \cosh k\gamma \cosh k\theta_>) \\
&= -2\sinh k\theta_< \cosh k(\gamma-\theta_>) \ .
\end{aligned}$$

We thus have that

$$G = \int_0^\infty \frac{4\sinh k\theta_< \cosh k(\gamma-\theta_>) \sin k\rho' \sin k\rho}{k\cosh k\gamma}\, dk \ ,$$

or, since $2\sin k\rho' \sin k\rho = \cos k(\rho'-\rho) - \cos k(\rho'+\rho)$,

$$G = \int_0^\infty \left[\cos k(\rho_> - \rho_<) - \cos k(\rho'+\rho)\right] \frac{2\sinh k\theta_< \cosh k(\gamma-\theta_>)}{k\cosh k\gamma}\, dk \ . \tag{3.15}$$

In order to obtain Green's function in closed form, we need to perform the integral above, which is the difference between two integrals of the type of the integral I that is calculated below. Such calculation is based on an usual application of the residue theorem: We express the integral (of an even function) as half its extension to the whole real axis, close its path with a semicircle C_R^+ of radius $R \to \infty$ in the upper half of the complex plane of $z = x + \mathrm{i}y$, notice that, according to Jordan's lemma {cf. Ref. [1, eq. (7.43)]}, the integral over C_R^+ tends to zero, and evaluate the residues at the simple poles of the integrand inside the closed contour $C = [-R,R] \cup C_R^+$, that is, at the zeros $n\pi\mathrm{i}/2\gamma$ $(n = 1,3,5,\cdots)$ of $\cosh\gamma z$ ($z = 0$ is a removable singularity).

$$\begin{aligned}
I &\equiv \int_0^\infty \cos Ax \frac{2\sinh\alpha x \cosh\beta x}{x\cosh\gamma x}\, dx \;=\; \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^\infty e^{\mathrm{i}Ax}\frac{2\sinh\alpha x \cosh\beta x}{x\cosh\gamma x}\, dx \\
&= \frac{1}{2}\,\mathrm{Re}\left(\oint_C - \overbrace{\int_{C_R^+}}^{\nearrow 0}\right) e^{\mathrm{i}Az}\frac{2\sinh\alpha z \cosh\beta z}{z\cosh\gamma z}\, dz \;=\; \frac{1}{2}\,\mathrm{Re}\left[2\pi\mathrm{i}\sum_{n=1,3,5\cdots}\mathrm{Res}\left(\frac{n\pi\mathrm{i}}{2\gamma}\right)\right] \\
&= \pi\,\mathrm{Re}\left[\ \mathrm{i}\sum_{n=1,3,5\cdots}\ \lim_{z\to\frac{n\pi\mathrm{i}}{2\gamma}} \frac{z - \frac{n\pi\mathrm{i}}{2\gamma}}{\cosh\gamma z}\, e^{\mathrm{i}Az}\frac{2\sinh\alpha z \cosh\beta z}{z}\right]
\end{aligned}$$

$$= \pi \,\mathrm{Re}\left[\ \mathrm{i}\sum_{n=1,3,5\cdots} \frac{1}{\gamma \mathrm{i} \sin\frac{n\pi}{2}}\, e^{-\frac{An\pi}{2\gamma}}\, \frac{2\cdot \mathrm{i}\sin\frac{\alpha n\pi}{2\gamma}\cos\frac{\beta n\pi}{2\gamma}}{\frac{n\pi \mathrm{i}}{2\gamma}}\right] ,$$

$$= \sum_{n=1,3,5\cdots} \frac{4\,e^{-\frac{n\pi A}{2\gamma}}}{n\sin(n\pi/2)} \sin\frac{n\pi\alpha}{2\gamma}\cos\frac{n\pi\beta}{2\gamma}\ , \tag{3.16}$$

with $A \geq 0$, $\gamma > 0$, and $\alpha+\beta \leq \gamma$. These restrictions upon the parameters A, α, β, and γ guarantee the convergence of the integral and the use of Jordan's lemma.

Taking (3.16) into account, we can develop (3.15) as follows:

$$G = I\Big|_{A=\rho_>-\rho_<} - I\Big|_{A=\rho'+\rho} \qquad \big(\text{with } \alpha=\theta_< \text{ and } \beta=\gamma-\theta_>\big)$$

$$= \sum_{n=1,3,5\cdots} \frac{4}{n}\, \frac{e^{-\frac{n\pi(\rho_>-\rho_<)}{2\gamma}} - e^{-\frac{n\pi(\rho'+\rho)}{2\gamma}}}{\sin(n\pi/2)} \sin\frac{n\pi\theta_<}{2\gamma}\cos\frac{n\pi(\gamma-\theta_>)}{2\gamma}\ .$$

At this point, we make use of (3.1) to return to the original variables r a r', obtaining

$$G = \sum_{n=1,3,5\cdots} \frac{4}{n}\, \frac{e^{-\frac{n\pi}{2\gamma}\left(\ln\frac{b}{r_<}-\ln\frac{b}{r_>}\right)} - e^{-\frac{n\pi}{2\gamma}\left(\ln\frac{b}{r'}+\ln\frac{b}{r}\right)}}{\cancel{\sin\frac{n\pi}{2}}} \sin\frac{n\pi\theta_<}{2\gamma}\,\cancel{\sin\frac{n\pi}{2}}\,\sin\frac{n\pi\theta_>}{2\gamma}$$

$$= \sum_{n=1,3,5\cdots} \frac{4}{n}\left[\left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{2\gamma}} - \left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{2\gamma}}\right] \sin\frac{n\pi\theta'}{2\gamma}\sin\frac{n\pi\theta}{2\gamma}\ . \tag{3.17}$$

We see that we got rid of the integral in (3.15), but now having to find the sum of the infinite series in (3.17) to express Green's function in closed form; we calculate this sum in Section 4. By the way, (2.14) and (3.17) give exactly the same expression for the desired Green's function.

4 GREEN'S FUNCTION IN CLOSED FORM

To derive Green's function in closed form we need to calculate the sum of the infinite series in (3.17). To this end, using the complex variable $z \equiv p e^{\mathrm{i}\varphi}$ ($p \in \mathbb{R}$ and $\varphi \in \mathbb{R}$), from which $z^n = p^n e^{\mathrm{i}n\varphi} = p^n \cos n\varphi + \mathrm{i} p^n \sin n\varphi$, we first evaluate the sum of the following infinite series:

$$\sum_{n=1,3,5\cdots} \frac{1}{n} \underbrace{p^n\cos n\varphi}_{\mathrm{Re}\, z^n} = \mathrm{Re}\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1} = \mathrm{Re}\int_0^z \left[\sum_{k=0}^{\infty} (\zeta^2)^k\right] d\zeta$$

$$= \mathrm{Re}\int_0^z \frac{1}{1-\zeta^2}\, d\zeta = \mathrm{Re}\left[\frac{1}{2}\ln(z+1) - \frac{1}{2}\ln(z-1)\right]$$

$$= \frac{1}{2}\ln|z+1| - \frac{1}{2}\ln|z-1| = \frac{1}{4}\ln\frac{1+p^2+2p\cos\varphi}{1+p^2-2p\cos\varphi} \quad (0 \leq p < 1)\ ,$$

where, to find the sum of the infinite series between square brackets, we used the formula for the sum of a geometric series (noticing that $|\zeta^2| \leq |z|^2 = p^2 < 1$ along the straight path of integration from $\zeta = 0$ to $\zeta = z$), and we also considered the definition of the complex logarithm.

Now, using the result above, we deduce the sum of this other infinite series:

$$
\begin{aligned}
S \equiv & \sum_{n=1,3,5\cdots} \frac{4}{n}\left(\frac{A}{B}\right)^{\frac{n\pi}{2\gamma}} \sin\frac{n\pi\theta'}{2\gamma}\sin\frac{n\pi\theta}{2\gamma} \\
= & \sum_{n=1,3,5\cdots} \frac{2}{n} p^n \left[\cos\tfrac{n\pi(\theta'-\theta)}{2\gamma} - \cos\tfrac{n\pi(\theta'+\theta)}{2\gamma}\right]\Bigg|_{p\,=\,\left(\frac{A}{B}\right)^{\pi/2\gamma}} \\
= & \left[\frac{1}{2}\ln\frac{1+p^2+2p\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{1+p^2-2p\cos\frac{\pi(\theta'-\theta)}{2\gamma}} - \frac{1}{2}\ln\frac{1+p^2+2p\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{1+p^2-2p\cos\frac{\pi(\theta'+\theta)}{2\gamma}}\right]_{p\,=\,\left(\frac{A}{B}\right)^{\pi/2\gamma}} \\
= & \frac{1}{2}\ln\frac{A^{\frac{\pi}{\gamma}}+B^{\frac{\pi}{\gamma}}+2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{A^{\frac{\pi}{\gamma}}+B^{\frac{\pi}{\gamma}}-2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}} - \frac{1}{2}\ln\frac{A^{\frac{\pi}{\gamma}}+B^{\frac{\pi}{\gamma}}+2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{A^{\frac{\pi}{\gamma}}+B^{\frac{\pi}{\gamma}}-2(AB)^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}} \;.
\end{aligned}
$$

Finally, since

$$
G(r,\theta\,|\,r',\theta') \;=\; S\Big|_{\substack{A=r_<\\ B=r_>}} \;-\; S\Big|_{\substack{A=rr'/b\\ B=b}} \quad ,
$$

as we see from (3.17), we have that

$$
\begin{aligned}
2\,G(r,\theta|r',\theta') = & \ln\frac{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}} - \ln\frac{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{r^{\frac{\pi}{\gamma}}+r'^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}} \\
& - \ln\frac{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}}{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'-\theta)}{2\gamma}} + \ln\frac{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}+2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}}{(rr'/b)^{\frac{\pi}{\gamma}}+b^{\frac{\pi}{\gamma}}-2(rr')^{\frac{\pi}{2\gamma}}\cos\frac{\pi(\theta'+\theta)}{2\gamma}} \;. \qquad (4.1)
\end{aligned}
$$

5 COMPARISON WITH THE SOLUTION GIVEN BY THE METHOD OF IMAGES WHEN THE DOMAIN IS THE FIRST QUADRANT OF THE DISC

The method of images {cf. Refs. [13, Sec. VII.13] and [6, Sec. 3]} allows to obtain the solution faster when the problem presents a symmetry that allows to quickly infer the configuration of images to use. Let us then apply this method for the particular case in which $\gamma = \pi/2$ to check (4.1). In this case, we need the seven images P_1^-, P_2^+, P_3^-, P_4^-, P_5^+, P_6^-, and P_7^+ shown in Figure 3, where, in this notation, the superscript $^+$ or $^-$ indicates that, in Green's function expression, the corresponding harmonic term is added or subtracted. These terms are of the form [13, Sec. VII.13, last paragraph] $\pm\ln(1/|\mathbf{r}'-\mathbf{r}_n|)$, if the corresponding image is generated by reflection with respect to the x or y-axis, or $\pm\ln\left[(b/r)/|\mathbf{r}'-\mathbf{r}_n|\right]$, if by inversion with respect to the circle of radius b centered at the origin, where $\mathbf{r}_n$ denotes the position vetor of the n-th image. Green's function is therefore given by

$$G(\mathbf{r}\,|\,\mathbf{r}') = \ln\frac{1}{|\mathbf{r}'-\mathbf{r}|} - \ln\frac{1}{|\mathbf{r}'-\mathbf{r}_1|} + \ln\frac{1}{|\mathbf{r}'-\mathbf{r}_2|} - \ln\frac{b/r}{|\mathbf{r}'-\mathbf{r}_3|}$$
$$- \ln\frac{1}{|\mathbf{r}'-\mathbf{r}_4|} + \ln\frac{b/r}{|\mathbf{r}'-\mathbf{r}_5|} - \ln\frac{b/r}{|\mathbf{r}'-\mathbf{r}_6|} + \ln\frac{b/r}{|\mathbf{r}'-\mathbf{r}_7|} \; . \tag{5.1}$$

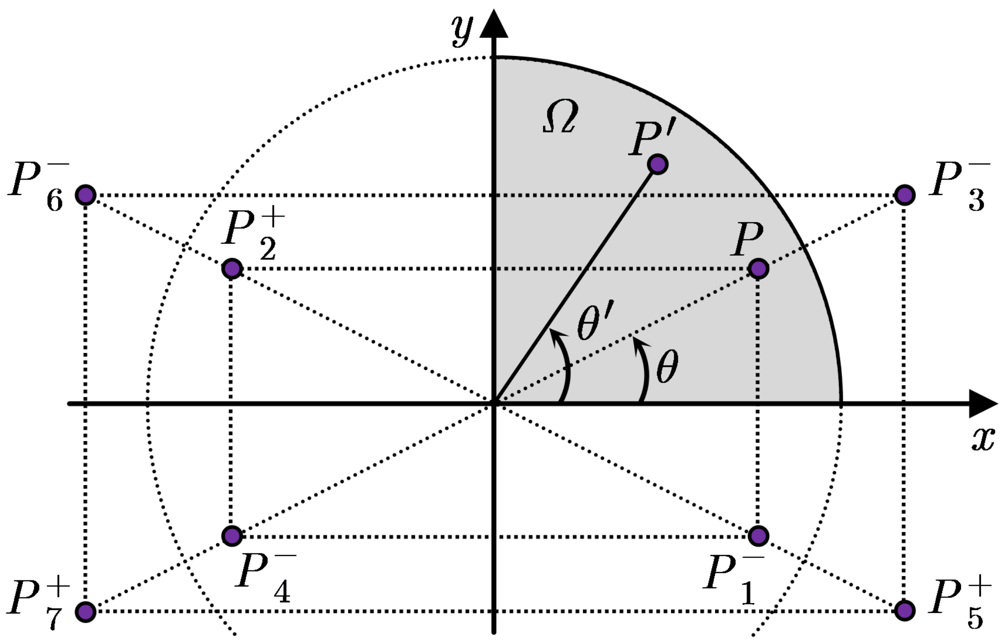

Figure 3: The configuration of images used to get, by the method of images, Green's function for problem (1.1) when $\gamma = \pi/2$ (the case in which the domain Ω is the first quadrant of the disc).

The plane polar coordinates of the position vectors above are as follows:

$$\mathbf{r}\,(r,\,\theta)\,,\quad \mathbf{r}'(r',\,\theta')\,,\quad \mathbf{r}_1(r,\,2\pi-\theta)\,,\quad \mathbf{r}_2(r,\,\pi-\theta)\,,\quad \mathbf{r}_3(b^2/r,\,\theta)\,,$$
$$\mathbf{r}_4(r,\,\pi+\theta)\,,\quad \mathbf{r}_5(b^2/r,\,2\pi-\theta)\,,\quad \mathbf{r}_6(b^2/r,\,\pi-\theta)\,,\quad \mathbf{r}_7(b^2/r,\,\pi+\theta)\;.$$

Therefore, by using the definition of magnitude of a vector, or, geometrically, the law of cosines, we can calculate all the distances $|\mathbf{r}'-\mathbf{r}|$, $|\mathbf{r}'-\mathbf{r}_1|$, $|\mathbf{r}'-\mathbf{r}_2|\cdots$ in (5.1), obtaining

$$G(r,\theta|r',\theta') = -\frac{1}{2}\ln\left[r^2 + r'^2 - 2rr'\cos(\theta'-\theta)\right] + \frac{1}{2}\ln\left[r^2 + r'^2 - 2rr'\cos(\theta'+\theta)\right]$$
$$-\frac{1}{2}\ln\left[r^2 + r'^2 + 2rr'\cos(\theta'+\theta)\right] + \frac{1}{2}\ln\left[\left(\frac{rr'}{b}\right)^2 + b^2 - 2rr'\cos(\theta'-\theta)\right]$$

$$+\frac{1}{2}\ln\left[r^2+r'^2+2rr'\cos(\theta'-\theta)\right]-\frac{1}{2}\ln\left[\left(\frac{rr'}{b}\right)^2+b^2-2rr'\cos(\theta'+\theta)\right]$$

$$+\frac{1}{2}\ln\left[\left(\frac{rr'}{b}\right)^2+b^2+2rr'\cos(\theta'+\theta)\right]-\frac{1}{2}\ln\left[\left(\frac{rr'}{b}\right)^2+b^2+2rr'\cos(\theta'-\theta)\right],$$

which is exactly the same Green's function given by (4.1) with $\gamma=\pi/2$.

6 RESULTS FOR ONE MORE PROBLEM

In this section we present the results for an extra problem so that we can observe how the calculations associated with the method exposed here vary. This presentation is not detailed like the one above, showing only the main intermediate results and the final one. We only show the intermediate results obtained with angular eigenfunctions (Section 2), omitting those related to radial eigenfunctions (Section 3) due to lack of space.

Let us consider Problem (1.1) with one modification: the replacement of Dirichlet's condition in (1.1b) with the Neumann's condition $[\partial u/\partial\theta](r,0)=g_0(r)$. In this case, according to Section 2, the angular eigenfunctions to be used are $\Theta_n(\theta)=\cos(n\pi\theta/\gamma)$ $(n=0,1,2,3,\cdots)$ {cf. Ref. [2], Sec. 10.1, Prob. 18, p. 595 & 786}, and, performing the calculations, we verify that the equations of that Section that are indicated below become as shown:

- Equation (2.1), for the modified problem of this section, becomes

$$v(r,\theta)=\frac{1}{2}v_0(r)+\sum_{n=1}^{\infty}v_n(r)\cos\frac{n\pi\theta}{\gamma}\ .$$

- Equation (2.2) :

$$v_n''+\frac{1}{r}v_n'-\frac{(n\pi/\gamma)^2}{r^2}v_n(r)\ =\ \frac{2}{\gamma}\int_0^{\gamma}h(r,\theta)\cos\frac{n\pi\theta}{\gamma}\,d\theta\ \equiv\ h_n(r)\quad(n\geq 0)\,.$$

- Equation (2.3) :

$$v_{P0}(r)=A_0+B_0\ln r\quad\text{and}\quad v_{Pn}(r)\Big|_{n\geq 1}=A_n(r)\,r^{n\pi/\gamma}+B_n(r)/r^{n\pi/\gamma}\,.$$

- Equation (2.8) :

$$v_0(r)\ =\ \int_0^b dr'r'h_0(r')\ln\frac{r'}{b}+\int_0^r dr'r'h_0(r')\ln\frac{r}{r'}\ ;$$

$$v_n(r)\Big|_{n\geq 1}\ =\ \frac{\gamma}{2n\pi}\int_0^b dr'r'h_n(r')\left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{\gamma}}-\left(\frac{r}{r'}\right)^{\frac{n\pi}{\gamma}}\right]$$

$$+\frac{\gamma}{2n\pi}\int_0^r dr'r'h_n(r')\left[\left(\frac{r}{r'}\right)^{\frac{n\pi}{\gamma}}-\left(\frac{r'}{r}\right)^{\frac{n\pi}{\gamma}}\right].$$

• Equation (2.10) :

$$
\begin{aligned}
v_0(r) &= \int_0^b dr' r' h_0(r') \ln\frac{r}{b} + \int_r^b dr' r' h_0(r') \ln\frac{r'}{r} \ ; \\
v_n(r)\Big|_{n\geq 1} &= \frac{\gamma}{2n\pi}\int_0^b dr' r' h_n(r') \left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{\gamma}} - \left(\frac{r'}{r}\right)^{\frac{n\pi}{\gamma}}\right] \\
&\quad + \frac{\gamma}{2n\pi}\int_r^b dr' r' h_n(r') \left[\left(\frac{r'}{r}\right)^{\frac{n\pi}{\gamma}} - \left(\frac{r}{r'}\right)^{\frac{n\pi}{\gamma}}\right] .
\end{aligned}
$$

• Equation (2.12) :

$$
\begin{aligned}
v_0(r) &= \int_0^b dr' r' h_0(r') \ln\frac{r_>}{b} \ ; \\
v_n(r)\Big|_{n\geq 1} &= \frac{\gamma}{2n\pi}\int_0^b dr' r' h_n(r') \left[\left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{\gamma}} - \left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{\gamma}}\right] .
\end{aligned}
$$

• Equation (2.14) :

$$
G(r,\theta\,|\,r',\theta') = \frac{2\pi}{\gamma}\ln\frac{b}{r_>} + \sum_{n=1}^{\infty}\frac{2}{n}\left[\left(\frac{r_<}{r_>}\right)^{\frac{n\pi}{\gamma}} - \left(\frac{rr'}{b^2}\right)^{\frac{n\pi}{\gamma}}\right]\cos\frac{n\pi\theta'}{\gamma}\cos\frac{n\pi\theta}{\gamma} \ .
$$

To express this result in closed form, we work in a similar way to that in Section 4, first obtaining the following formula:

$$
\sum_{n=1}^{\infty}\frac{1}{n}p^n\cos n\varphi = -\frac{1}{2}\ln(1+p^2-2p\cos\varphi) \quad (0\leq p<1) .
$$

Then, using it, we get the sum of this other infinite series:

$$
\begin{aligned}
S \equiv & \sum_{n=1}^{\infty}\frac{2}{n}\left(\frac{A}{B}\right)^{\frac{n\pi}{\gamma}}\cos\frac{n\pi\theta'}{\gamma}\cos\frac{n\pi\theta}{\gamma} \\
&= -\frac{1}{2}\ln\left[1+\left(\frac{A}{B}\right)^{\frac{2\pi}{\gamma}} - 2\left(\frac{A}{B}\right)^{\frac{\pi}{\gamma}}\cos\frac{\pi(\theta'-\theta)}{\gamma}\right] \\
&\quad -\frac{1}{2}\ln\left[1+\left(\frac{A}{B}\right)^{\frac{2\pi}{\gamma}} - 2\left(\frac{A}{B}\right)^{\frac{\pi}{\gamma}}\cos\frac{\pi(\theta'+\theta)}{\gamma}\right] .
\end{aligned}
$$

Finally, since

$$
G(r,\theta\,|\,r',\theta') = \frac{2\pi}{\gamma}\ln\frac{b}{r_>} + S\Big|_{\substack{A=r_< \\ B=r_>}} - S\Big|_{\substack{A=rr'/b \\ B=b}} \ ,
$$

we deduce the following closed form for Green's function:

$$\begin{aligned} G(r,\theta\,|\,r',\theta') = \frac{2\pi}{\gamma}\ln\frac{b}{r_>} &-\frac{1}{2}\ln\left[1+\left(\frac{r_<}{r_>}\right)^{\frac{2\pi}{\gamma}}-2\left(\frac{r_<}{r_>}\right)^{\frac{\pi}{\gamma}}\cos\frac{\pi(\theta'-\theta)}{\gamma}\right] \\ &-\frac{1}{2}\ln\left[1+\left(\frac{r_<}{r_>}\right)^{\frac{2\pi}{\gamma}}-2\left(\frac{r_<}{r_>}\right)^{\frac{\pi}{\gamma}}\cos\frac{\pi(\theta'+\theta)}{\gamma}\right] \\ &+\frac{1}{2}\ln\left[1+\left(\frac{rr'}{b^2}\right)^{\frac{2\pi}{\gamma}}-2\left(\frac{rr'}{b^2}\right)^{\frac{\pi}{\gamma}}\cos\frac{\pi(\theta'-\theta)}{\gamma}\right] \\ &+\frac{1}{2}\ln\left[1+\left(\frac{rr'}{b^2}\right)^{\frac{2\pi}{\gamma}}-2\left(\frac{rr'}{b^2}\right)^{\frac{\pi}{\gamma}}\cos\frac{\pi(\theta'+\theta)}{\gamma}\right] . \end{aligned}$$

To allow a comparison of this result with the one provided by the method of images, we consider the particular case of a semi-disc, that is, we let $\gamma=\pi$ above, obtaining

$$\begin{aligned} G(r,\theta\,|\,r',\theta') &= -\frac{1}{2}\ln\left[r^2+r'^2-2rr'\cos(\theta'-\theta)\right]-\frac{1}{2}\left[r^2+r'^2-2rr'\cos(\theta'+\theta)\right] \\ &+\frac{1}{2}\ln\left[b^2+\left(\frac{rr'}{b}\right)^2-2rr'\cos(\theta'-\theta)\right]+\frac{1}{2}\ln\left[b^2+\left(\frac{rr'}{b}\right)^2-2rr'\cos(\theta'+\theta)\right] . \end{aligned}$$

Let us build Green's function using the method of imagens, taking Section 5 into account. Figure 4 shows the three required images P_1^+, P_2^-, and P_3^+. This configuration of images implies the following Green's function:

$$G(\mathbf{r}\,|\,\mathbf{r}') = \ln\frac{1}{|\mathbf{r}'-\mathbf{r}|}+\ln\frac{1}{|\mathbf{r}'-\mathbf{r}_1|}-\ln\frac{b/r}{|\mathbf{r}'-\mathbf{r}_2|}-\ln\frac{b/r}{|\mathbf{r}'-\mathbf{r}_3|} \;,$$

the development of which leads to the same expression in the previous equation.

7 FINAL COMMENTS

The method can be applied in any domain Ω where $r\in(a,b)$ and $\theta\in(0,\gamma)$, with any a and b such that $b>a\geq 0$ and any $\gamma\in(0,2\pi]$. Furthermore, many experimental calculations performed privately indicate that, whenever the boundary conditions are Dirichlet's or Neumann's, it will be possible to determine Green's function in *closed form*, which is a nice feature of the method.

When there are enough symmetries (what may happen when $\gamma=\pi/2,\,\pi,\,3\pi/2,\,2\pi$), the method of images also gives results in closed form and, moreover, more quickly, thus becoming the best method. But when this is not the case, this method becomes considerably involved, and the method presented here is preferable.

Consider the problem in Figure 1 in the particular case of a semi-disc, whose closed-form Green's function is given by (4.1) with $\gamma=\pi$. This is an interesting example of a problem in which using the method of images is very complicated. The complication lies in the boundary conditions; indeed, when such conditions are those in Figure 4, the application of this method is simple.

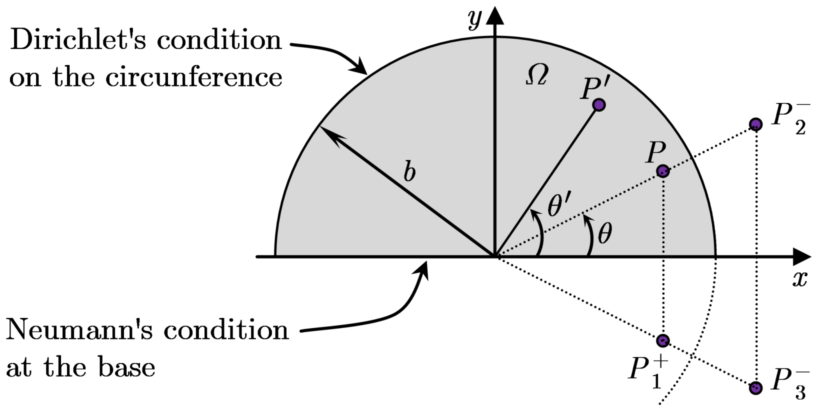

Figure 4: The configuration of images to get Green's function for the problem considered in Section 6 when $\gamma = \pi$.

It is expected that the method presented in this work will find application in problems that are objects of research. Let us mention a few. It could be applied to calculate the gravitational potential in spiral galaxies (like the Milky Way) that typically have a flattened, disc-like structure. In this problem, the surface mass density forming the inhomogeneous term of Poisson's equation could be given by eq. (2) in Ref. [8]. In addition, at a specific radius representing the edge of the observable galaxy, the gravitational potential or the flux of matter across it would compose Dirichlet's or Neumann's condition, respectively (depending on the available measured data). Other applications would be the calculation of the electric field at two-dimensional corners and along sharp edges (cf. [10, Sec. 2.11], [11], and [3]), or the calculation of the solution to the incompressible Navier-Stokes equations [12, eq. (15)] in the case of two-dimensional flows on domains shaped like circular sectors, in which it is required the solution to the pressure Poisson equation under Neumann's conditions [12, eq. (16)].

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