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Sums of generalized third-order Jacobsthal numbers by matrix methods

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ABSTRACT. In this paper, we consider a certain third-order linear recurrence and then give generating matrices for the sums of positively and negatively subscripted terms of this recurrence. Further, we use matrix methods and derive explicit formulas for these sums.

Keywords: recurrence, sum, matrix method, companion matrix, third-order Jacobsthal number.

1 INTRODUCTION

For $n \geq 1$, the third-order Jacobsthal sequence is defined by the following relation:

$$\mathscr{J}_{n+2}^{(3)} = \mathscr{J}_{n+1}^{(3)} + \mathscr{J}_{n}^{(3)} + 2\mathscr{J}_{n-1}^{(3)},$$

where $\mathscr{J}_{0}^{(3)} = 0$ and $\mathscr{J}_{1}^{(3)} = \mathscr{J}_{2}^{(3)} = 1$ (see, e.g. [1,9]). The third-order Jacobsthal numbers have many interesting properties [3,7,8,10]. For example, the sums of the third-order Jacobsthal numbers subscripted from 1 to n can be expressed by a formula including third-order Jacobsthal numbers. The sums formula is given by

$$\sum_{s=1}^{n} \mathscr{J}_{s}^{(3)} = \frac{1}{3}\left(\mathscr{J}_{n+2}^{(3)} + 2\mathscr{J}_{n}^{(3)} - 1\right).$$

On the other hand, matrix methods many times have played an important role stemming from the number theory [2,5]. Some applications of this topic can be reviewed in the following literature [4,6], and several of the references cited in these works. For instance, let $\mathscr{J}$ be an 3×3 companion matrix as follows

$$\mathscr{J} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

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Then it is well known that $\mathscr{J}_{-1}^{(3)} = 0$ and

$$\mathscr{J}^n = \begin{bmatrix} \mathscr{J}_{n+1}^{(3)} & \mathscr{J}_{n+2}^{(3)} - \mathscr{J}_{n+1}^{(3)} & 2\mathscr{J}_{n}^{(3)} \\ \mathscr{J}_{n}^{(3)} & \mathscr{J}_{n+1}^{(3)} - \mathscr{J}_{n}^{(3)} & 2\mathscr{J}_{n-1}^{(3)} \\ \mathscr{J}_{n-1}^{(3)} & \mathscr{J}_{n}^{(3)} - \mathscr{J}_{n-1}^{(3)} & 2\mathscr{J}_{n-2}^{(3)} \end{bmatrix} \ (n \geq 1).$$

Now, we consider a generalization of the third-order Jacobsthal numbers. Let k be nonzero integer satisfying $k^2 - 4 \neq 0$. The generalized third-order Jacobsthal sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ is defined by the recurrence relation for $n \geq 1$:

$$\mathscr{J}_{k,n+2}^{(3)} = (2-k)\mathscr{J}_{k,n+1}^{(3)} + (2k-1)\mathscr{J}_{k,n}^{(3)} + 2\mathscr{J}_{k,n-1}^{(3)}, \tag{1.1}$$

where $\mathscr{J}_{k,0}^{(3)} = 0$, $\mathscr{J}_{k,1}^{(3)} = 1$ and $\mathscr{J}_{k,2}^{(3)} = 2-k$.

For later use, note that $\mathscr{J}_{k,3}^{(3)} = 3 - 2k + k^2$ and $\mathscr{J}_{k,4}^{(3)} = 6 - 2k + 2k^2 - k^3$. When $k = 1$, then $\mathscr{J}_{k,n}^{(3)} = \mathscr{J}_{n}^{(3)}$ (n-th third-order Jacobsthal number).

Let ω_1 and ω_2 be the roots of the equation $x^2 + kx + 1 = 0$, then the Binet formula of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ has the form

$$\mathscr{J}_{k,n}^{(3)} = \frac{1}{5+2k}\left[2^{n+1} - \left(\frac{\alpha(k)\omega_1^n - \beta(k)\omega_2^n}{\omega_1 - \omega_2}\right)\right],$$

where $\alpha(k) = -1 + 2\omega_1$ and $\beta(k) = -1 + 2\omega_2$.

Using the recurrence relation of sequence $\{\mathscr{J}_{k,n}^{(3)}\}$, we can obtain the negatively subscripted terms and these terms satisfy

$$\mathscr{J}_{k,-n}^{(3)} = \frac{1}{5+2k}\left[2^{-n+1} - \left(\frac{\alpha(k)\omega_1^{-n} - \beta(k)\omega_2^{-n}}{\omega_1 - \omega_2}\right)\right].$$

Since $\omega_1 + \omega_2 = -k$ and $\omega_1\omega_2 = 1$, then we have

$$\mathscr{J}_{k,-(n+3)}^{(3)} = \frac{1}{2}\left[(1-2k)\mathscr{J}_{k,-(n+2)}^{(3)} + (k-2)\mathscr{J}_{k,-(n+1)}^{(3)} + \mathscr{J}_{k,-n}^{(3)}\right]. \tag{1.2}$$

Thus for later use $\mathscr{J}_{k,-2}^{(3)} = \frac{1}{2}$, $\mathscr{J}_{k,-3}^{(3)} = \frac{1}{4}(1-2k)$ and $\mathscr{J}_{k,-4}^{(3)} = -\frac{1}{8}(3+2k-4k^2)$.

Furthermore, by the inductive argument, one can easily verify that the generating matrix for the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ is given by

$$\mathscr{J}_k^n = \begin{bmatrix} 2-k & 2k-1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} \mathscr{J}_{k,n+1}^{(3)} & \mathscr{J}_{k,n+2}^{(3)} + (k-2)\mathscr{J}_{k,n+1}^{(3)} & 2\mathscr{J}_{k,n}^{(3)} \\ \mathscr{J}_{k,n}^{(3)} & \mathscr{J}_{k,n+1}^{(3)} + (k-2)\mathscr{J}_{k,n}^{(3)} & 2\mathscr{J}_{k,n-1}^{(3)} \\ \mathscr{J}_{k,n-1}^{(3)} & \mathscr{J}_{k,n}^{(3)} + (k-2)\mathscr{J}_{k,n-1}^{(3)} & 2\mathscr{J}_{k,n-2}^{(3)} \end{bmatrix}. \tag{1.3}$$

In this paper, we construct certain matrices, then we compute the n-th powers of these matrices which are the generating matrices for the sums of the positively and negatively subscripted terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ from 1 to n.

2 GENERATING MATRIX FOR THE SUMS OF THE POSITIVELY SUBSCRIPTED TERMS OF THE SEQUENCE $\{\mathscr{J}_{K,N}^{(3)}\}$

In this section, we consider the positively subscripted terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ and then define a 4×4 matrix $\mathscr{G}$. Further, we compute the n-th power of the matrix $\mathscr{G}$ and use matrix methods for the explicit formula for the sums of terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$.

Define the 4×4 matrix $\mathscr{G}$ as follows

$$\mathscr{G}=\begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 2-k & 2k-1 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.1}$$

and define the 4×4 matrix $\mathscr{G}_k(n)$ as follows

$$\mathscr{G}_k(n)=\begin{bmatrix} 1 & 0 & 0 & 0\\ \mathscr{S}_{k,n}^{(+)} & \mathscr{J}_{k,n+1}^{(3)} & \mathscr{J}_{k,n+2}^{(3)}+(k-2)\mathscr{J}_{k,n+1}^{(3)} & 2\mathscr{J}_{k,n}^{(3)}\\ \mathscr{S}_{k,n-1}^{(+)} & \mathscr{J}_{k,n}^{(3)} & \mathscr{J}_{k,n+1}^{(3)}+(k-2)\mathscr{J}_{k,n}^{(3)} & 2\mathscr{J}_{k,n-1}^{(3)}\\ \mathscr{S}_{k,n-2}^{(+)} & \mathscr{J}_{k,n-1}^{(3)} & \mathscr{J}_{k,n}^{(3)}+(k-2)\mathscr{J}_{k,n-1}^{(3)} & 2\mathscr{J}_{k,n-2}^{(3)} \end{bmatrix}, \tag{2.2}$$

where $\mathscr{S}_{k,n}^{(+)}$ denote the sums of the positively subscripted terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ from 1 to n, that is

$$\mathscr{S}_{k,n}^{(+)}=\sum_{s=1}^{n}\mathscr{J}_{k,s}^{(3)}. \tag{2.3}$$

Then, we have the following result.

Lemma 1. *Let $\mathscr{G}$ and $\mathscr{G}_k(n)$ be matrices of the form (2.1) and (2.2), respectively. Then, for $n\geq 1$*

$$\mathscr{G}_k(n)=\mathscr{G}^n. \tag{2.4}$$

Proof. We will use the induction method for the proof of the lemma. If $n=1$, then, by $\mathscr{J}_{k,-1}^{(3)}=\mathscr{J}_{k,0}^{(3)}=0$, $\mathscr{J}_{k,1}^{(3)}=1$ and $\mathscr{J}_{k,2}^{(3)}=2-k$, we obtain

$$\mathscr{G}^1=\begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 2-k & 2k-1 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0\\ \mathscr{S}_{k,1}^{(+)} & \mathscr{J}_{k,2}^{(3)} & \mathscr{J}_{k,3}^{(3)}+(k-2)\mathscr{J}_{k,2}^{(3)} & 2\mathscr{J}_{k,1}^{(3)}\\ \mathscr{S}_{k,0}^{(+)} & \mathscr{J}_{k,1}^{(3)} & \mathscr{J}_{k,2}^{(3)}+(k-2)\mathscr{J}_{k,1}^{(3)} & 2\mathscr{J}_{k,0}^{(3)}\\ \mathscr{S}_{k,-1}^{(+)} & \mathscr{J}_{k,0}^{(3)} & \mathscr{J}_{k,1}^{(3)}+(k-2)\mathscr{J}_{k,0}^{(3)} & 2\mathscr{J}_{k,-1}^{(3)} \end{bmatrix}=\mathscr{G}_k(1).$$

Suppose that the claim is true for n. Then, we will show that the equation hold for $n+1$. Thus, by our assumption, we write

$$
\begin{aligned}
\mathscr{G}^{n+1} &= \mathscr{G}^n \cdot \mathscr{G}^1 \\
&= \mathscr{G}_k(n) \cdot \mathscr{G}^1 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathscr{S}^{(+)}_{k,n} & \mathscr{J}^{(3)}_{k,n+1} & \mathscr{J}^{(3)}_{k,n+2} + (k-2)\mathscr{J}^{(3)}_{k,n+1} & 2\mathscr{J}^{(3)}_{k,n} \\ \mathscr{S}^{(+)}_{k,n-1} & \mathscr{J}^{(3)}_{k,n} & \mathscr{J}^{(3)}_{k,n+1} + (k-2)\mathscr{J}^{(3)}_{k,n} & 2\mathscr{J}^{(3)}_{k,n-1} \\ \mathscr{S}^{(+)}_{k,n-2} & \mathscr{J}^{(3)}_{k,n-1} & \mathscr{J}^{(3)}_{k,n} + (k-2)\mathscr{J}^{(3)}_{k,n-1} & 2\mathscr{J}^{(3)}_{k,n-2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2-k & 2k-1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

which, by a matrix multiplication, satisfies

$$
\mathscr{G}^{n+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathscr{S}^{(+)}_{k,n} + \mathscr{J}^{(3)}_{k,n+1} & \mathscr{J}^{(3)}_{k,n+2} & \mathscr{J}^{(3)}_{k,n+3} + (k-2)\mathscr{J}^{(3)}_{k,n+2} & 2\mathscr{J}^{(3)}_{k,n+1} \\ \mathscr{S}^{(+)}_{k,n-1} + \mathscr{J}^{(3)}_{k,n} & \mathscr{J}^{(3)}_{k,n+1} & \mathscr{J}^{(3)}_{k,n+2} + (k-2)\mathscr{J}^{(3)}_{k,n+1} & 2\mathscr{J}^{(3)}_{k,n} \\ \mathscr{S}^{(+)}_{k,n-2} + \mathscr{J}^{(3)}_{k,n-1} & \mathscr{J}^{(3)}_{k,n} & \mathscr{J}^{(3)}_{k,n+1} + (k-2)\mathscr{J}^{(3)}_{k,n} & 2\mathscr{J}^{(3)}_{k,n-1} \end{bmatrix}.
$$

By the recurrence relation of the sequence $\{\mathscr{J}^{(3)}_{k,n}\}$ and since $\mathscr{S}^{(+)}_{k,n} + \mathscr{J}^{(3)}_{k,n+1} = \mathscr{S}^{(+)}_{k,n+1}$, we have the conclusion. □

Consequently, we obtain a generating matrix for the sums of the terms of the generalized third-order Jacobsthal sequence from 1 to n.

Also we write the Eq. (2.4) as shown

$$
\mathscr{G}_k(n+1) = \mathscr{G}_k(n)\mathscr{G}_k(1) = \mathscr{G}_k(1)\mathscr{G}_k(n). \tag{2.5}
$$

In other words, the matrix $\mathscr{G}_k(1)$ is commutative under matrix multiplication. Then, we have the next result.

Corollary 2. *Let the sum $\mathscr{S}^{(+)}_{k,n}$ have the form (2.3). Then, the sum $\mathscr{S}^{(+)}_{k,n}$ satisfies the following non-homogeneous recurrence relation for $n \geq 1$*

$$
\mathscr{S}^{(+)}_{k,n+3} = (2-k)\mathscr{S}^{(+)}_{k,n+2} + (2k-1)\mathscr{S}^{(+)}_{k,n+1} + 2\mathscr{S}^{(+)}_{k,n} + 1.
$$

Proof. From Eq. (2.5) and since an element of $\mathscr{G}_k(n+1)$ is the product of a row $\mathscr{G}_k(1)$ and a column of $\mathscr{G}_k(n)$:

$$
\mathscr{S}^{(+)}_{k,n+1} = (2-k)\mathscr{S}^{(+)}_{k,n} + (2k-1)\mathscr{S}^{(+)}_{k,n-1} + 2\mathscr{S}^{(+)}_{k,n-2} + 1,
$$

which is desired. □

Now we are going to derive an explicit formula for the sum $\mathscr{S}_{k,n}^{(+)}$ with the generalized third-order Jacobsthal numbers. Let $\mathscr{K}_{\mathscr{G}}(\lambda)$ be the characteristic polynomial of the matrix $\mathscr{G}$. Thus, we have

$$\mathscr{K}_{\mathscr{G}}(\lambda)=\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & (2-k)-\lambda & 2k-1 & 2 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix}=\lambda^4+(k-3)\lambda^3+3(1-k)\lambda^2+(2k-3)\lambda+2.$$

Also it is easily seen that the characteristic polynomial of the matrix $\mathscr{J}_k$ given by Eq. (1.3) is $\lambda^3+(2-k)\lambda^2+(2k-1)\lambda+2$. Therefore the eigenvalues of the matrix $\mathscr{G}$ are

$$\lambda_1=\omega_1=\frac{-k+\sqrt{k^2-4}}{2},\ \lambda_2=\omega_2=\frac{-k-\sqrt{k^2-4}}{2},\ \lambda_3=2,\ \lambda_4=1.$$

Since $k\neq 0$ and $k^2-4\neq 0$, we have that the eigenvalues of the matrix $\mathscr{G}$ are distinct.

Then, we have the following result.

Theorem 3. *Let $\mathscr{S}_{k,n}^{(+)}$ denote the sums of the terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$. Then,*

$$\mathscr{S}_{k,n}^{(+)}=\frac{1}{k+2}\left(\mathscr{J}_{k,n+2}^{(3)}+(k-1)\mathscr{J}_{k,n+1}^{(3)}+2\mathscr{J}_{k,n}^{(3)}-1\right). \tag{2.6}$$

Proof. Let $\mathscr{L}$ be the 4×4 matrix defined as follows:

$$\mathscr{L}=\begin{bmatrix} -k-2 & 0 & 0 & 0 \\ 1 & 4 & -\omega_1 k-1 & -\omega_2 k-1 \\ 1 & 2 & \omega_1 & \omega_2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tag{2.7}$$

where ω_1,ω_2 are the eigenvalues of $\mathscr{J}_k$. Note that $\det(\mathscr{L})=(k+2)(2\omega_2-1)(\omega_1-\omega_2+2)\neq 0$. One can easily verify that $\mathscr{G}\mathscr{L}=\mathscr{L}\mathscr{D}$, where $\mathscr{G}$ and $\mathscr{L}$ are as before, and $\mathscr{D}$ is the diagonal matrix such that $\mathscr{D}=diag(\lambda_4,\lambda_3,\lambda_1,\lambda_2)$. Since $\det(\mathscr{L})\neq 0$, the matrix $\mathscr{L}$ is invertible. So, we write that $\mathscr{D}=\mathscr{L}^{-1}\mathscr{G}\mathscr{L}$.

Hence, the matrix $\mathscr{G}$ is similar to the diagonal matrix $\mathscr{D}$. Thus, we obtain $\mathscr{G}^n\mathscr{L}=\mathscr{L}\mathscr{D}^n$. Since that $\mathscr{G}_k(n)=\mathscr{G}^n$, we have $\mathscr{G}_k(n)\mathscr{L}=\mathscr{L}\mathscr{D}^n$. Then, the coefficient 1 in the second row and first column of $\mathscr{L}\mathscr{D}^n$ corresponds to

$$\begin{bmatrix} \mathscr{S}_{k,n}^{(+)} & \mathscr{J}_{k,n+1}^{(3)} & \mathscr{J}_{k,n+2}^{(3)}+(k-2)\mathscr{J}_{k,n+1}^{(3)} & 2\mathscr{J}_{k,n}^{(3)} \end{bmatrix}\begin{bmatrix} -k-2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

So by a matrix multiplication, we have the conclusion. □

For example, if we take $k=1$, then the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ is reduced to the usual third-order Jacobsthal numbers and we obtain

$$\sum_{s=1}^{n} \mathscr{J}_s^{(3)} = \frac{1}{3}\left(\mathscr{J}_{n+2}^{(3)} + 2\mathscr{J}_n^{(3)} - 1\right)$$

which is well-known.

Now, we give a formula for the sum $\mathscr{S}_{k,n}^{(+)}$ by using a matrix method with the following result.

Corollary 4. *Let $\mathscr{S}_{k,n}^{(+)}$ denote the sums of the terms $\mathscr{J}_{k,i}^{(3)}$ from 1 to n. Then, for all positive integers n and r, we have*

$$\mathscr{S}_{k,n+r}^{(+)} = \mathscr{S}_{k,n}^{(+)} + \mathscr{J}_{k,n+1}^{(3)}\mathscr{S}_{k,r}^{(+)} + \left(\mathscr{J}_{k,n+2}^{(3)} + (k-2)\mathscr{J}_{k,n+1}^{(3)}\right)\mathscr{S}_{k,r-1}^{(+)} + 2\mathscr{J}_{k,n}^{(3)}\mathscr{S}_{k,r-2}^{(+)},$$

where $\mathscr{J}_{k,n}^{(3)}$ given by Eq. (1.1).

Proof. From Eq. (2.4) in Lemma 1, we can write, for all positive integers n and r, $\mathscr{G}_k(n+r) = \mathscr{G}_k(n)\mathscr{G}_k(r)$. Then, the coefficient $\mathscr{S}_{k,n+r}^{(+)}$ in the second row and first column of $\mathscr{G}_k(n+r)$ corresponds to

$$\begin{bmatrix} \mathscr{S}_{k,n}^{(+)} & \mathscr{J}_{k,n+1}^{(3)} & \mathscr{J}_{k,n+2}^{(3)} + (k-2)\mathscr{J}_{k,n+1}^{(3)} & 2\mathscr{J}_{k,n}^{(3)} \end{bmatrix} \begin{bmatrix} 1 \\ \mathscr{S}_{k,r}^{(+)} \\ \mathscr{S}_{k,r-1}^{(+)} \\ \mathscr{S}_{k,r-2}^{(+)} \end{bmatrix}.$$

By a matrix multiplication, the proof is easily seen. □

Note that taking $n=1$ in Corollary 4, we can obtain the result of Corollary 2.

3 GENERATING MATRIX FOR THE SUMS OF THE NEGATIVELY SUBSCRIPTED TERMS $\{\mathscr{J}_{K,-N}^{(3)}\}$

In this section, we consider the negatively subscripted terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$. First, we give a generating matrix for the negatively subscripted terms. Second, we give a generating matrix for the sums of these terms.

Let the 3×3 matrix $\mathscr{H}_k$ be as follows

$$\mathscr{H}_k = \begin{bmatrix} \frac{1}{2}(1-2k) & \frac{1}{2}(k-2) & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.1}$$

and the 3×3 matrix $\mathscr{H}(n)$ be as follows

$$\mathscr{H}(n) = \begin{bmatrix} 2\mathscr{J}_{k,-(n+2)}^{(3)} & \mathscr{J}_{k,-n}^{(3)} + (k-2)\mathscr{J}_{k,-(n+1)}^{(3)} & \mathscr{J}_{k,-(n+1)}^{(3)} \\ 2\mathscr{J}_{k,-(n+1)}^{(3)} & \mathscr{J}_{k,-(n-1)}^{(3)} + (k-2)\mathscr{J}_{k,-n}^{(3)} & \mathscr{J}_{k,-n}^{(3)} \\ 2\mathscr{J}_{k,-n}^{(3)} & \mathscr{J}_{k,-(n-2)}^{(3)} + (k-2)\mathscr{J}_{k,-(n-1)}^{(3)} & \mathscr{J}_{k,-(n-1)}^{(3)} \end{bmatrix}, \tag{3.2}$$

where $\mathscr{J}_{k,-n}^{(3)}$ es the n-th negatively subscripted term of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$.

We start with the following result.

Lemma 1. *Let $\mathscr{H}_k$ and $\mathscr{H}(n)$ be matrices of the form (3.1) and (3.2), respectively. Then, for all $n \geq 1$*

$$\mathscr{H}(n) = \mathscr{H}_k^n.$$

Proof. (Induction on n). If $n = 1$, then by identity in Eq. (1.2), we have

$$\mathscr{H}_k^1 = \begin{bmatrix} \frac{1}{2}(1-2k) & \frac{1}{2}(k-2) & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2\mathscr{J}_{k,-3}^{(3)} & \mathscr{J}_{k,-1}^{(3)} + (k-2)\mathscr{J}_{k,-2}^{(3)} & \mathscr{J}_{k,-2}^{(3)} \\ 2\mathscr{J}_{k,-2}^{(3)} & \mathscr{J}_{k,0}^{(3)} + (k-2)\mathscr{J}_{k,-1}^{(3)} & \mathscr{J}_{k,-1}^{(3)} \\ 2\mathscr{J}_{k,-1}^{(3)} & \mathscr{J}_{k,1}^{(3)} + (k-2)\mathscr{J}_{k,0}^{(3)} & \mathscr{J}_{k,0}^{(3)} \end{bmatrix}.$$

We suppose that the equation hold for n. Then, we will show that the equation holds for $n+1$. Thus, by our assumption,

$$\begin{aligned} \mathscr{H}_k^{n+1} &= \mathscr{H}_k^n \mathscr{H}_k^1 \\ &= \begin{bmatrix} 2\mathscr{J}_{k,-(n+2)}^{(3)} & \mathscr{J}_{k,-n}^{(3)} + (k-2)\mathscr{J}_{k,-(n+1)}^{(3)} & \mathscr{J}_{k,-(n+1)}^{(3)} \\ 2\mathscr{J}_{k,-(n+1)}^{(3)} & \mathscr{J}_{k,-(n-1)}^{(3)} + (k-2)\mathscr{J}_{k,-n}^{(3)} & \mathscr{J}_{k,-n}^{(3)} \\ 2\mathscr{J}_{k,-n}^{(3)} & \mathscr{J}_{k,-(n-2)}^{(3)} + (k-2)\mathscr{J}_{k,-(n-1)}^{(3)} & \mathscr{J}_{k,-(n-1)}^{(3)} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \frac{1}{2}(1-2k) & \frac{1}{2}(k-2) & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

From the negatively subscripted terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ satisfy the recurrence relation

$$\mathscr{J}_{k,-(n+3)}^{(3)} = \frac{1}{2}\left[(1-2k)\mathscr{J}_{k,-(n+2)}^{(3)} + (k-2)\mathscr{J}_{k,-(n+1)}^{(3)} + \mathscr{J}_{k,-n}^{(3)}\right].$$

and $\mathscr{H}(n+1) = \mathscr{H}_k^{n+1}$. So the proof is completed. □

Let $\mathscr{S}_{k,n}^{(-)}$ denote the sums of the negatively subscripted terms of the sequence $\{\mathscr{J}_{k,n}^{(3)}\}$ from 1 to n, that is

$$\mathscr{S}_{k,n}^{(-)} = \sum_{s=1}^{n} \mathscr{J}_{k,-s}^{(3)}. \tag{3.3}$$

Now, we give a matrix method to generate the sum $\mathscr{S}_{k,n}^{(-)}$. Define the 4×4 matrix $\mathscr{R}$ as follows

$$\mathscr{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}(1-2k) & \frac{1}{2}(k-2) & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.4}$$

and define the 4×4 matrix $\mathscr{R}_k(n)$ as follows

$$\mathscr{R}_k(n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathscr{S}^{(-)}_{k,n+1} & 2\mathscr{J}^{(3)}_{k,-(n+2)} & \mathscr{J}^{(3)}_{k,-n} + (k-2)\mathscr{J}^{(3)}_{k,-(n+1)} & \mathscr{J}^{(3)}_{k,-(n+1)} \\ \mathscr{S}^{(-)}_{k,n} & 2\mathscr{J}^{(3)}_{k,-(n+1)} & \mathscr{J}^{(3)}_{k,-(n-1)} + (k-2)\mathscr{J}^{(3)}_{k,-n} & \mathscr{J}^{(3)}_{k,-n} \\ \mathscr{S}^{(-)}_{k,n-1} & 2\mathscr{J}^{(3)}_{k,-n} & \mathscr{J}^{(3)}_{k,-(n-2)} + (k-2)\mathscr{J}^{(3)}_{k,-(n-1)} & \mathscr{J}^{(3)}_{k,-(n-1)} \end{bmatrix}. \tag{3.5}$$

Then, we have the following result.

Theorem 2. *Let $\mathscr{R}$ and $\mathscr{R}_k(n)$ be matrices of the form (3.4) and (3.5), respectively. Then, for all $n \geq 1$, we have*

$$\mathscr{R}^n = \mathscr{R}_k(n). \tag{3.6}$$

Proof. (Induction on n) If $n = 1$, then we know that $\mathscr{S}^{(-)}_{k,1} = \mathscr{J}^{(3)}_{k,-1} = 0$, $\mathscr{S}^{(-)}_{k,n} = 0$ for $n = -1$ and $\mathscr{J}^{(3)}_{k,-2} = \frac{1}{2}$. Thus we obtain $\mathscr{R}^1 = \mathscr{R}_k(1)$. Suppose that the equation holds for n. Then, we will show that the equation holds for $n+1$. Thus, by our assumption, we write

$$\begin{aligned} \mathscr{R}^{n+1} &= \mathscr{R}^n \mathscr{R}^1 \\ &= \mathscr{R}_k(n)\mathscr{R}^1 \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathscr{S}^{(-)}_{k,n+1} & 2\mathscr{J}^{(3)}_{k,-(n+2)} & \mathscr{J}^{(3)}_{k,-n} + (k-2)\mathscr{J}^{(3)}_{k,-(n+1)} & \mathscr{J}^{(3)}_{k,-(n+1)} \\ \mathscr{S}^{(-)}_{k,n} & 2\mathscr{J}^{(3)}_{k,-(n+1)} & \mathscr{J}^{(3)}_{k,-(n-1)} + (k-2)\mathscr{J}^{(3)}_{k,-n} & \mathscr{J}^{(3)}_{k,-n} \\ \mathscr{S}^{(-)}_{k,n-1} & 2\mathscr{J}^{(3)}_{k,-n} & \mathscr{J}^{(3)}_{k,-(n-2)} + (k-2)\mathscr{J}^{(3)}_{k,-(n-1)} & \mathscr{J}^{(3)}_{k,-(n-1)} \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}(1-2k) & \frac{1}{2}(k-2) & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Using $\mathscr{S}^{(-)}_{k,n} = \mathscr{S}^{(-)}_{k,n} + \mathscr{J}^{(3)}_{k,-(n+1)}$ and by Lemma 1, we obtain $\mathscr{R}^{n+1} = \mathscr{R}_k(n+1)$. So the proof is completed. □

In the following result, we give a non-homogeneous recurrence relation for the sum $\mathscr{S}^{(-)}_{k,n}$.

Theorem 3. *Let $\mathscr{S}^{(-)}_{k,n}$ denote the sums of the terms $\{\mathscr{J}^{(3)}_{k,-i}\}$ for $1 \leq i \leq n$. Then, for all $n \geq 1$, we have*

$$\mathscr{S}^{(-)}_{k,n+2} = \frac{1}{2}\left[(1-2k)\mathscr{S}^{(-)}_{k,n+1} + (k-2)\mathscr{S}^{(-)}_{k,n} + \mathscr{S}^{(-)}_{k,n-1} + 1\right].$$

Proof. Considering Eq. (3.6), we write $\mathscr{R}_k(n+1) = \mathscr{R}_k(n)\mathscr{R}_k(1) = \mathscr{R}_k(1)\mathscr{R}_k(n)$ and say that the matrix $\mathscr{R}_k(1)$ is commutative under matrix multiplication. By a matrix multiplication, the proof is easy. □

Generalizing $\mathscr{R}^n = \mathscr{R}_k(n)$, for all positive integers n and r, we can write that

$$\mathscr{R}_k(n+r) = \mathscr{R}_k(n)\mathscr{R}_k(r) = \mathscr{R}_k(r)\mathscr{R}_k(n).$$

Thus, we obtain the following corollary without proof as a generalization of the result of Theorem 3.

Corollary 4. *Let $\mathscr{S}_{k,n}^{(-)}$ denote the sums of the terms $\{\mathscr{J}_{k,-i}^{(3)}\}$ for $1 \leq i \leq n$. Then, for all $n, r \geq 1$, we have*

$$\mathscr{S}_{k,n+r+1}^{(-)} = \mathscr{S}_{k,n+1}^{(-)} + 2\mathscr{J}_{k,-(n+2)}^{(3)}\mathscr{S}_{k,r+1}^{(-)} + \left(\mathscr{J}_{k,-n}^{(3)} + (k-2)\mathscr{J}_{k,-(n+1)}^{(3)}\right)\mathscr{S}_{k,r}^{(-)} + \mathscr{J}_{k,-(n+1)}^{(3)}\mathscr{S}_{k,r-1}^{(-)}.$$

Now, we derive an explicit formula for the sums of the negatively subscripted terms $\mathscr{J}_{k,-i}^{(3)}$ for $1 \leq i \leq n$. For this purpose, we give some results. First, we consider the characteristic polynomial of the matrix $\mathscr{R}$. The characteristic equation of $\mathscr{R}$ is

$$\mathscr{K}_{\mathscr{R}}(\mu) = \mu^4 + \frac{1}{2}(2k-3)\mu^3 + \frac{3}{2}(1-k)\mu^2 + \frac{1}{2}(k-3)\mu + \frac{1}{2}.$$

Thus, the eigenvalues of matrix $\mathscr{R}$ are

$$\mu_1 = \omega_1 = \frac{-k+\sqrt{k^2-4}}{2},\ \mu_2 = \omega_2 = \frac{-k-\sqrt{k^2-4}}{2},\ \mu_3 = \frac{1}{2},\ \mu_4 = 1.$$

Note that $k \neq 0$ and $k^2 - 4 \neq 0$, the eigenvalues of $\mathscr{R}$ are distinct.

Then, we have the following result.

Theorem 5. *Let $\mathscr{S}_{k,n}^{(-)}$ denote the sums of the negatively subscripted terms $\mathscr{J}_{k,-i}^{(3)}$ for $1 \leq i \leq n$. Then, for all $n \geq 1$, we have*

$$\mathscr{S}_{k,n}^{(-)} = -\frac{1}{k+2}\left(2\mathscr{J}_{k,-(n+2)}^{(3)} + (2k+1)\mathscr{J}_{k,-(n+1)}^{(3)} + \mathscr{J}_{k,-n}^{(3)} - 1\right). \tag{3.7}$$

Proof. Let $\mathscr{M}$ be the 4×4 matrix defined as follows:

$$\mathscr{M} = \begin{bmatrix} k+2 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & -\omega_1 k - 1 & -\omega_2 k - 1 \\ 1 & \frac{1}{2} & \omega_1 & \omega_2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tag{3.8}$$

where ω_1, ω_2 are the eigenvalues of $\mathscr{J}_k$. Note that $\det(\mathscr{M}) = \frac{1}{4}(k+2)(2-\omega_2)(2(\omega_1-\omega_2)+1) \neq 0$. One can easily verify that $\mathscr{R}\mathscr{M} = \mathscr{M}\mathscr{D}$, where $\mathscr{R}$ and $\mathscr{M}$ are as before, and $\mathscr{D}$ is the diagonal matrix such that $\mathscr{D} = diag(\mu_4, \mu_3, \mu_1, \mu_2)$. Since $\det(\mathscr{M}) \neq 0$, the matrix $\mathscr{M}$ is invertible. So, we write that $\mathscr{D} = \mathscr{M}^{-1}\mathscr{R}\mathscr{M}$.

Hence, the matrix $\mathscr{R}$ is similar to the diagonal matrix $\mathscr{D}$. Thus, we obtain $\mathscr{R}^n\mathscr{M} = \mathscr{M}\mathscr{D}^n$. Since that $\mathscr{R}_k(n) = \mathscr{R}^n$, we have

$$\mathscr{R}_k(n)\mathscr{L} = \mathscr{L}\mathscr{D}^n.$$

So by a matrix multiplication, we have the conclusion. □

For example, if take $k=1$, then the sequence $\mathscr{J}_{k,n}^{(3)}$ is reduced to the usual third-order Jacobsthal sequence and by Theorem 5, we have the sums of the negatively subscripted terms of the third-order Jacobsthal sequence for $n \geq 1$,

$$\mathscr{S}_{k,n}^{(-)} = -\frac{1}{3}\left(2\mathscr{J}_{k,-(n+2)}^{(3)} + 3\mathscr{J}_{k,-(n+1)}^{(3)} + \mathscr{J}_{k,-n}^{(3)} - 1\right).$$

4 CONCLUSIONS

In this study, we defined the generalized third-order Jacobsthal numbers, which is an extension of the third-order Jacobsthal numbers. We provided the Binet formula and the generating matrices for the sums of positively and negatively subscripted terms of this recurrence. For future research, additional identities and generalizations of the generalized third-order Jacobsthal numbers can be studied.

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