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On the (*r,p,k*)-Generalized Jacobsthal Numbers and the Jacobsthal Fundamental Fibonacci System

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ABSTRACT. The main goal of this paper is to study the model of the sequences of (r,p,k)-generalized Jacobsthal numbers, by the approach based on the properties of its associated Jacobsthal fundamental Fibonacci system and its fundamental sequence. Some linear and combinatorial properties are established. Moreover, the related matrix formulation allows us to provide new identities, which are declined in combinatorial form. Especially, the linear Jacobsthal Cassini identity and its combinatorial formulation are furnished. Finally, illustrative special cases and examples are given.

Keywords: Jacobsthal Fundamental Fibonacci System, matrix representation, combinatorial formulas.

1 INTRODUCTION

The well known linear recurrence relation defining the sequence of usual Fibonacci numbers $\{F_n\}_{n\geq 0}$ is given by $F_{n+1} = F_n + F_{n-1}$, for $n \geq 1$, where $F_0 = 0$ and $F_1 = 1$. Recall that recursive formula of the sequence of usual Fibonacci numbers, established by De Moivre in the 19th century for describing mathematically the famous Fibonacci rabbit problem, represents the earliest and most prominent example of a recursive sequence and discrete dynamical system. Furthermore, recent extensive studies on this topic, although of a theoretical nature, have been motivated by its numerous applications in different mathematical and applied mathematics fields, as well as in the exact and applied sciences, and art (see, for example, [5] and references therein).

Subsequently, since the Fibonacci numbers were declined under the preceding recursive formula, several important sequences of numbers have been provided in the literature, which are defined by an analogous weighted linear recurrence relation, such as the Pell numbers, Padovan-Perrin numbers, Leonardo numbers and Jacobsthal numbers. Especially, the usual sequence of usual Jacobsthal numbers $\{J_n\}_{n\geq 0}$ is defined by the linear recurrence relation

$$J_{n+1} = J_n + 2J_{n-1} \quad \text{for} \quad n \geq 1, \tag{1.1}$$

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where $J_0 = 0$ and $J_1 = 1$. The sequences of usual Fibonacci and Jacobsthal numbers have been generalized in the literature under various formulations. For more details, the Jacobsthal numbers and some generalizations, can be seen in the references [11, 12], and references therein. Among these generalizations, we can find the two following recursive expressions, based on the order of recursiveness. The first one, is the sequence $\{J_n\}_{n\geq 0}$ defined by linear recurrence relation of order r

$$J_{n+1} = J_n + 2J_{n-1} + J_{n-2} + \cdots + J_{n-r+2} + J_{n-r+1}, \text{ for } n \geq r, \tag{1.2}$$

where $J_0 = \alpha_0, \cdots, J_{r-1} = \alpha_{r-1}$ are the arbitrary initial conditions. The second generalizations of the sequence of usual Jacobsthal numbers (1.1) is the sequence $\{J_n\}_{n\geq 0}$ defined by the linear recurrence relation of order r

$$J_{n+1} = J_n + J_{n-1} + J_{n-2} + \cdots + J_{n-r+2} + 2J_{n-r+1}, \text{ for } n \geq r, \tag{1.3}$$

where $J_0 = \alpha_0, \cdots, J_{r-1} = \alpha_{r-1}$ are the initial conditions.

Let $r \geq 2$, $k \geq 1$ and $p \geq 1$ be a given integers in $\mathbf{N}$. The two sequences of generalized Jacobsthal numbers defined by the linear recursive relation (1.2)-(1.3), can be extended to family of (r,p,k)-generalized Jacobsthal numbers $\{J_n\}_{n\geq 0}$ defined as follows

$$J_{n+1} = J_n + 2^p J_{n-1} + J_{n-2} + \cdots + J_{n-r+2} + 2^k J_{n-r+1}, \text{ for } n \geq r, \tag{1.4}$$

where $J_0 = \alpha_0, \cdots, J_{r-1} = \alpha_{r-1}$, are the initial conditions. In fact, the two families of generalized Jacobsthal numbers defined by Expressions (1.2)- (1.3), represent a special class of the model of the (r,p,k)-generalized Jacobsthal numbers (1.4). Indeed, we can observe that for $p = 1$ and $k = 0$ in (1.4)(respectively, $p = 0, k = 1$), we get the generalized Jacobsthal numbers (1.2) (respectively, (1.3)) In addition, for $p = k = 0$ Expression (1.4) is reduced to the well known generalized Fibonacci numbers of order r, studied in various research papers.

This paper aims to study some properties of the (r,p,k)-generalized Jacobsthal numbers defined by expression (1.4), and its special cases (1.2)- (1.3). Our approach is based on the so-called the fundamental Fibonacci system and its related fundamental sequence. This approach permits us to establish some linear and combinatorial properties of the (r,p,k)-generalized Jacobsthal numbers. In addition, the matrix formulation of Expression (1.4) allows us to provide some new identities, especially the Cassini identity and its combinatorial expression. Moreover, similar properties of the special cases (1.2)- (1.3) are derived.

The content of this study is organized as follows. Section 2 concerns some properties of the vector space of the sequences of (r,p,k)-generalized Jacobsthal numbers, and its related Jacobsthal fundamental Fibonacci system. Especially, the fundamental sequence is introduced and its main role is presented. In Section 3 properties of the matrix formulation of Expression (1.4) is studied in terms of its related Jacobsthal fundamental Fibonacci system. In addition, some new identities associated to the (r,p,k)-generalized Jacobsthal numbers are provided, in terms of the fundamental sequence. Section 4 is devoted to the combinatorial expression of the sequence of (r,p,k)-generalized Jacobsthal numbers and its related identities. More precisely, using the combinatorial formulation of the fundamental sequence and the results of sections 2 and 3, we obtain the combinatorial form of the sequence of (r,p,k)-generalized Jacobsthal numbers and its related identities. In Section 5, we provide some properties of the Cassini Identity associated to the sequence of (r,p,k)-generalized Jacobsthal numbers. Finally, the special cases (1.2)- (1.3) are considered and examples are furnished.

2 THE VECTOR SPACE OF (r,p,k)-GENERALIZED JACOBSTHAL NUMBERS AND ITS FUNDAMENTAL FIBONACCI SYSTEM

2.1 The generalized Jacobsthal vector space and its fundamental Fibonacci system

Let $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$, where $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$, be the set of sequences (r,p,k)-generalized Jacobsthal numbers of order r defined by the recurrence relation (1.4), with arbitrary initial conditions $(\alpha_0,\alpha_1,\cdots,\alpha_{r-1})\in\mathbf{K}^r$. Equipped with the usual addition and a multiplication by a scalar, we can show that $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$ is $\mathbf{K}$-vector space. Let $\{\{J_n^{(s)}\}_{n\geq 0};\ 0\leq s\leq r-1\}$ be the family of sequences defined by (1.4), with mutually different sets of initial values, indexed by s $(0\leq s\leq r-1)$, defined as follows

$$J_{n+1}^{(s)}=J_n^{(s)}+2^pJ_{n-1}^{(s)}+J_{n-2}^{(s)}+\cdots+J_{n-r+2}^{(s)}+2^kJ_{n-r+1}^{(s)},\quad \text{for } n\geq r-1, \tag{2.1}$$

where $J_n^{(s)}=\delta_{s,n}$ $(0\leq n\leq r-1)$ are the initial, here $\delta_{s,n}$ is the Kronecker symbol, namely, $\delta_{s,s}=1$ and $\delta_{s,n}=0$, for $s\neq n$. Let $\beta_0,\cdots,\ \beta_{r-1}$ be scalars of $\mathbf{K}$, and suppose that $\sum_{s=0}^{r-1}\beta_sJ_n^{(s)}=0$, for $n\geq 0$. Then, for $n=0,1,\ldots,r-1$, we have $\sum\limits_{s=0}^{r-1}\beta_sJ_n^{(s)}=\beta_nJ_n^{(n)}=\beta_n=0$, because $J_n^{(s)}=0$ for $n\neq s$ and $J_n^{(n)}=1$. Thus, the set of sequences (2.1) is a linearly independent system of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$. In addition, let $\{J_n\}_{n\geq 0}$ be a sequence in $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$, with initial conditions $\alpha_0,\cdots,\alpha_{r-1}$. Then, we can verify that $J_n=\sum\limits_{s=0}^{r-1}\alpha_sJ_n^{(s)}$, for every $n\geq 0$. Indeed, let $\{w_n\}_{n\geq 0}$ be the sequence defined by $w_n=\sum_{s=0}^{r-1}\alpha_sJ_n^{(s)}$, for every $n\geq 0$. Then, clearly $\{w_n\}_{n\geq 0}$ is an element of $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$ and for $n=0,1,\ldots,r-1$ we have $w_n=\sum\limits_{s=0}^{r-1}\alpha_sJ_n^{(s)}=\alpha_nJ_n^{(n)}=\alpha_n$, for every n $(0\leq n\leq r-1.)$. Hence, the two sequences $\{J_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ satisfy Expression (2.1) and own the same initial conditions. Therefore, we have $J_n=w_n$, for every $n\geq 0$, which implies that the set of sequences (2.1) represents a generator system of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$. In summary, we can formulate the following proposition.

Proposition 1. *The set of sequences* $\{J_n^{(s)}\}_{n\geq 0}$ *(*$0\leq s\leq r-1$*) represents a basis of the* $\mathbf{K}$*-vector space* $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$ *(*$\mathbf{K}=\mathbf{R}$ *or* $\mathbf{C}$*), of the sequences of* (r,p,k)*-generalized Jacobsthal numbers. Moreover, we have* $\dim_{\mathbf{K}}\mathscr{E}_{\mathbf{K}}^{(r)}=r$.

The set $\{\{J_n^{(s)}\}_{n\geq 0};\ 0\leq s\leq r-1\}$ *is the called the Jacobsthal Fundamental Fibonacci System.* The Jacobsthal fundamental Fibonacci system will play a central role in the sequel of this study. In addition, the remarkable sequence $\{J_n^{(r-1)}\}_{n\geq 0}$ will allow us to obtain important results on the sequences of (r,p,k)-generalized Jacobsthal numbers and their $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$.

2.2 The Jacobsthal Fibonacci fundamental sequence

In this subsection, we study the closed relation between the sequence of (r,p,k)-generalized Jacobsthal numbers $\{J_n^{(r-1)}\}_{n\geq 0}$ and the other elementary sequences of (r,p,k)-generalized Jacobsthal numbers $\{J_n^{(s)}\}_{n\geq 0}$, where $0\leq s\leq r-1$. Moreover, we derive the compact formula of every element of the **K**-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$ in terms of the sequence $\{J_n^{(r-1)}\}_{n\geq 0}$.

To this aim, let first establish the expression of $\{J_n^{(0)}\}_{n\geq 0}$ in terms of $\{J_n^{(r-1)}\}_{n\geq 0}$.

Lemma 1. *Let $\{J_n^{(0)}\}_{n\geq 0}$ and $\{J_n^{(r-1)}\}_{n\geq 0}$ be the (r,p,k)-generalized Jacobsthal numbers sequences (2.1). Then, for each $n\geq 1$, we have,*

$$J_n^{(0)} = 2^k J_{n-1}^{(r-1)}. \tag{2.2}$$

Proof. We proceed by induction on n. For $n=1$, we have $J_1^{(0)}=\cdots=J_{r-1}^{(0)}=0$ and $J_{r-1}^{(0)}=2^k$. On the other hand, $J_0^{(r-1)}=J_1^{(r-1)}=\cdots=J_{r-2}^{(r-1)}=0$ and $J_{r-1}^{(r-1)}=1$. Thus, we have $J_d^{(0)}=0=2^k.0=2^k.J_{d-1}^{(r-1)}$, for $d=1,\cdots,r-1$ and $J_r^{(0)}=2^k=2^k.1=2^kJ_{r-1}^{(r-1)}$. Therefore, $J_n^{(0)}=2^kJ_{n-1}^{(r-1)}$ for $1\leq n\leq r-1$. Suppose that $J_n^{(0)}=2^kJ_{n-1}^{(r-1)}$, for some $n\geq r$. Let establish that $J_{n+1}^{(0)}=2^kJ_n^{(r-1)}$. We have $J_{n+1}^{(0)}=J_n^{(0)}+2^pJ_{n-1}^{(0)}+J_{n-2}^{(0)}+\cdots+2^kJ_{n-r+1}^{(0)}$, for $n\geq r-1$. Then, using the hypothesis of induction $J_n^{(0)}=2^kJ_{n-1}^{(r-1)}$, we obtain $J_{n+1}^{(0)}=2^kJ_{n-1}^{(r-1)}+2^k2^pJ_{n-2}^{(r-1)}+2^kJ_{n-3}^{(r-1)}+\cdots+2^k2^kJ_{n-r}^{(r-1)}$ or equivalently $2^k(J_{n-1}^{(r-1)}+2^pJ_{n-2}^{(r-1)}+J_{n-3}^{(r-1)}+\cdots+2^kJ_{n-r}^{(r-1)})=2^kJ_n^{(r-1)}$. Therefore, we get $J_n^{(0)}=2^kJ_{n-1}^{(r-1)}$, for all $n\geq 1$. □

Second, let now express the sequence $\{J_n^{(s)}\}_{n\geq 0}$, for $1\leq s\leq r-2$ in terms of the sequence $\{J_n^{(r-1)}\}_{n\geq 0}$.

Lemma 2. *Let $\{J_n^{(s)}\}_{n\geq 0}$, where $1\leq s\leq r-2$ and $\{J_n^{(r-1)}\}_{n\geq 0}$ be the generalized Jacobsthal sequences (2.1). Then, for $1\leq s\leq r-3$, we have,*

$$J_n^{(s)} = J_{n-1}^{(r-1)} + J_{n-2}^{(r-1)} + \cdots + J_{n-s}^{(r-1)} + 2^kJ_{n-s-1}^{(r-1)}, \tag{2.3}$$

for every $n\geq r$, and for $s=r-2$, we have

$$J_n^{(r-2)} = 2^pJ_{n-1}^{(r-1)} + J_{n-2}^{(r-1)} + \cdots + J_{n-r+2}^{(r-1)} + 2^kJ_{n-r+1}^{(r-1)}. \tag{2.4}$$

Proof. The proof is obtained by mathematical induction. For reason of convenience and simplicity, we set $a_0=a_2=\cdots=a_{r-2}=1$, $a_1=2^p$ and $a_{r-1}=2^k$. Let $\{J_n^{(s)}\}_{n\geq 0}$ $(1\leq s\leq r-2)$ be an element of the fundamental Fibonacci system of the **K**-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$, where $J_0^{(s)}=J_1^{(s)}=\ldots=J_{s-2}^{(s)}=J_{s-1}^{(s)}=\ldots=J_{r-1}^{(s)}=0$ and $J_s^{(s)}=1$. Then, by using the recursive formula (2.1), we remark that $J_r^{(s)}=a_{r-s-1}J_{r-1}^{(r-1)}$ and $J_{r+1}^{(s)}=a_{r-s-1}J_r^{(r-1)}+a_{r-s-2}J_{r-1}^{(r-1)}$. Continuing this process, we obtain $J_{2r-1}^{(s)}=a_{r-s-1}J_{2r-2}^{(r-1)}+a_{r-s-2}J_{2r-1}^{(r-1)}+\ldots+a_{r-1}J_{r-1}^{(r-1)}$. And the recurrence relation (2.1) (or more generally (1.4)) permits to get,

$$J_n^{(s)} = a_{r-s-1}J_{n-1}^{(r-1)} + a_{r-s}J_{n-2}^{(r-1)} + \ldots + a_{r-1}J_{n-s-1}^{(r-1)}. \tag{2.5}$$

Since $a_0 = a_2 = \cdots = a_{r-2} = 1$, $a_1 = 2^p$ and $a_{r-1} = 2^k$, we show that for $1 \leq s \leq r-3$, Expression (2.5) takes the form, $J_n^{(s)} = J_{n-1}^{(r-1)} + J_{n-2}^{(r-1)} + \cdots + J_{n-s}^{(r-1)} + 2^k J_{n-s-1}^{(r-1)}$, for every $n \geq r-1$, which is nothing else but Expression (2.3). Finally, for $s = r-2$, we derive that Expression (2.5) can be written under the form,

$$J_n^{(s)} = 2^p J_{n-1}^{(r-1)} + J_{n-2}^{(r-1)} + \cdots + J_{n-s}^{(r-1)} + 2^k J_{n-s-1}^{(r-1)}.$$

This last expression is none other than Expression (2.4). □

Expressions (2.2), (2.3) and (2.4) show that every element of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$ can be expressed only in terms of the (r,p,k)-generalized Jocobsthal sequence $\{J_n^{(r-1)}\}_{n\geq 0}$. More precisely, we have the following result.

Proposition 2. *Let $\{J_n\}_{n\geq 0}$ be a sequence of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$, of initial data $\alpha_0, \cdots, \alpha_{r-1}$. Then, for every $n \geq 0$, the general term J_n is given under the form,*

$$J_n = \sum_{j=0}^{r-1} Z_j J_{n-j-1}^{(r-1)}, \tag{2.6}$$

where

$$\begin{cases} Z_0 = \alpha_0 2^k + \alpha_1 + \cdots + \alpha_{r-3} + \alpha_{r-2} 2^p + \alpha_{r-1} \\ Z_1 = \alpha_1 2^k + \alpha_2 + \cdots + \alpha_{r-2} + \alpha_{r-1} 2^p \\ Z_j = \alpha_j 2^k + \alpha_{j+1} + \cdots + \alpha_{r-1}, \textit{ for } 2 \leq j \leq r-1. \end{cases} \tag{2.7}$$

Proof. As in the proof of Lemma 2, for reason of convenience and simplicity, we set $a_0 = a_2 = \cdots = a_{r-2} = 1$, $a_1 = 2^p$ and $a_{r-1} = 2^k$. Let $\{J_n\}_{n\geq 0}$ be a sequence of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$, of initial data $\alpha_0, \cdots, \alpha_{r-1}$. Proposition 1, concerning the linear decomposition in the fundamental Fibonacci system of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$, implies that we have $J_n = \sum_{s=0}^{r-1} \alpha_s J_n^{(s)} = \sum_{s=0}^{r-2} \alpha_s J_n^{(s)} + \alpha_{r-1} J_n^{(r-1)}$, for every $n \geq 0$. Whence, we substitute $J_n^{(s)}$ $(s = 0, ..., s = r-1)$, as shown in Expressions (2.2), (2.3) and (2.4), in the general terms J_n, we obtain $J_n = \sum_{s=0}^{r-1} \alpha_s \sum_{k=0}^{s} a_{r-s+k-1} J_{n-k-1}^{(r-1)} = \sum_{s=0}^{r-2} \alpha_s \sum_{k=0}^{s} a_{r-s+k-1} J_{n-k-1}^{(r-1)} + \alpha_{r-1} 2^k J_n^{(r-1)}$. The preceding equality can be expanded as follows,

$$\begin{cases} \text{For } s = 0 : \alpha_0 J_n^{(0)} = \alpha_0 a_{r-1} J_{n-1}^{(r-1)} \\ \text{For } s = 1 : \alpha_1 J_n^{(1)} = \alpha_1 a_{r-2} J_{n-1}^{(r-1)} + \alpha_1 a_{r-1} J_{n-2}^{(r-1)} \\ \vdots \\ \text{For } s = r-2 : \alpha_{r-2} J_n^{(r-1)} = \alpha_{r-2} a_1 J_{n-1}^{(r-1)} + \alpha_{r-2} a_2 J_{n-2}^{(r-1)} + ... + \alpha_{r-2} a_{r-1} J_{n-r+1}^{(r-1)} \\ \text{For } s = r-1 : \alpha_{r-1} J_n^{(r-1)} = \alpha_{r-1} a_0 J_{n-1}^{(r-1)} + \alpha_{r-1} a_1 J_{n-2}^{(r-1)} + ... + \alpha_{r-1} a_{r-1} J_{n-r}^{(r-1)}. \end{cases}$$

Therefore, we derive that the general terms J_n can be written under the form $J_n = \sum_{k=0}^{r-1} Z_k J_{n-k-1}^{(r-1)}$,

where $\begin{cases} Z_0 = \alpha_0 a_{r-1} + \alpha_1 a_{r-2} + \cdots + \alpha_{r-1} a_0 \\ Z_1 = \alpha_1 a_{r-1} + \alpha_2 a_{r-2} + \cdots + \alpha_{r-1} a_1 \\ \vdots \\ Z_{r-2} = \alpha_{r-2} a_{r-1} + \alpha_{r-1} a_{r-2} \text{ and } Z_{r-1} = \alpha_{r-1} a_{r-1}. \end{cases}$

So, we get Expressions (2.6)-(2.7). □

It reveals from Lemmas 1, 2 and Propositions 2, that we have the following corollary.

Corollary 1. *The family of sequences $\mathscr{J} = \{\{J_{n-s}^{(r-1)}\}_{n\geq s};\ s = 0,1,...,r-1\}$, is a basis of the* $\mathbf{K}$*-vector space* $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$.

Proof. Expression (2.6) shows that the set $\mathscr{J} = \{\{J_{n-s}^{(r-1)}\}_{n\geq s}; s = 0,1,...,r-1\}$ is a generating system of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$. Since $\mathscr{J}$ is of cardinal r and $\dim_{\mathbf{K}}(\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k) = r$, we derive that set $\mathscr{J} = \{\{J_{n-s}^{(r-1)}\}_{n\geq s}; s = 0,1,...,r-1\}$ is a basis of the $\mathbf{K}$-vector space $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$. □

Lemmas 1-2, Propositions 2 and Corollary 1 show that the sequence $\{J_n^{(r-1)}\}_{n\geq 0}$, will play an important role in the sequel. In analogy with the fundamental solution for ordinary differential equations of constant coefficients, we can formulate the following definition.

Definition 1. *The sequence $\{J_n^{(r-1)}\}_{n\geq 0}$ is called the fundamental sequence of the linear recursive Expression (1.4), or of the* $\mathbf{K}$*-vector space* $\mathscr{E}_{\mathbf{K}}^{(r)}(a_0,\ldots,a_{r-1})$.

The fundamental Fibonacci system and its related fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$ will play a central role in additive number theory. For the two special cases $k = 0$ or $p = 0$ of Expression (1.4), we get the following corollary.

Corollary 2. *Let $\{J_n\}$ be a sequence of the* $\mathbf{K}$*-vector space* $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$*, of initial data $\alpha_0,\cdots,\ \alpha_{r-1}$. Then, for $k = 0$ the general term J_n is written under the form $J_n = \sum_{j=0}^{r-1} Z_{p,j} J_{n-j-1}^{(r-1)}$, where the $Z_{p,j}$ are given by,*

$$\begin{cases} Z_{p,0} = \alpha_0 + \alpha_1 + \cdots + \alpha_{r-3} + \alpha_{r-2}2^p + \alpha_{r-1} \\ Z_{p,1} = \alpha_1 + \alpha_2 + \cdots + \alpha_{r-2} + \alpha_{r-1}2^p \\ Z_{p,j} = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{r-1},\ \textit{for}\ 2 \leq j \leq r-1. \end{cases} \tag{2.8}$$

And for $p = 0$, the general term J_n is given by $J_n = \sum_{j=0}^{r-1} Z_{k,j} J_{n-j-1}^{(r-1)}$, where the $Z_{k,j}$ are given by,

$$Z_{k,j} = \alpha_j 2^k + \sum_{d=j+1}^{r-1} \alpha_d \ \textit{for}\ 0 \leq j \leq r-1. \tag{2.9}$$

Remark 1. *For $k=0$ and $p=1$ or $k=1$ and $p=0$ (respectively), Corollary 2 allows us to derive the same properties for the generalized Jacobsthal numbers defined by (1.2)- (1.3), respectively.*

3 MATRIX FORMULATION OF THE (*r,p,k*)-GENERALIZED JACOBSTHAL NUMBERS AND SOME IDENTITIES

3.1 Matrix formulation of the sequence of (*r,p,k*)-generalized Jacobsthal numbers

For reason of clarity, we recall here the general setting of a sequence $\{v_n\}_{n\geq 0}$ defined by a linear recurrence relation $v_n=\sum_{i=0}^{r-1} a_i v_{n-i-1}$, for $n\geq r$, with arbitrary initial conditions $v_0,\ \dots,\ v_{r-1}$. The analogous fundamental Fibonacci system $\{(v_n^{(s)})_{n\geq 0},\ 0\leq s\leq r-1\}$ is defined as follows,

$$\begin{cases} v_n^{(s)}=\sum_{i=0}^{r-1} a_i v_{n-i-1}^{(s)} \qquad \text{for} \qquad n\geq r, \\ v_n^{(s)}=\delta_{s,n} \text{ for } 0\leq n\leq r-1. \end{cases} \tag{3.1}$$

Let $\mathbf{Y}_n$ be the vector column $\mathbf{Y}_n=(v_n,\ \dots,\ v_{n-r+1})^t$. The sequence $\{v_n\}_{n\geq 0}$ can be expressed under the following equivalent matrix form, $\mathbf{Y}_{n+1}=\mathbf{A}\mathbf{Y}_n$, for $n\geq r-1$, where $\mathbf{A}$ is the companion matrix $\mathbf{A}=\begin{bmatrix} a_0 & a_1 & & \cdots & a_{r-1} \\ 1 & 0 & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$, denoted in the sequel under the form $\mathbf{A}=\mathbf{A}[a_0,a_1,\cdots,a_{r-1}]$. Furthermore, an iterative process allows us to show that the matrix formulation of $\{v_n\}_{n\geq 0}$, can be written under the form, $\mathbf{Y}_{n+r-1}=\mathbf{A}^n\mathbf{Y}_{r-1}$, for every $n\geq 0$. For the matrix power $\mathbf{A}^n=(a_{ij}^{(n)})_{1\leq i,j\leq r}$ it was established in [3] (see also [2, 10]) that the entries $a_{ij}^{(n)}$ can be expressed in terms of the elements of the fundamental Fibonacci system (3.1). More precisely, the following powerful theorem is established.

Theorem 1 (see [3]). *Under the preceding data, the entries $a_{ij}^{(n)}$ of the powers $\mathbf{A}^n=(a_{ij}^{(n)})_{0\leq i,j\leq r-1}$ are given by,*

$$a_{ij}^{(n)}=v_{n+r-i-1}^{(r-j-1)} \tag{3.2}$$

where the sequences $\{v_n^{(s)}\}_{n\geq 0}$ $(0\leq s\leq r-1)$ are defined by (3.1). In other words, for every $n\geq 0$, we have,

$$\mathbf{A}^n=\begin{bmatrix} v_{n+r-1}^{(r-1)} & \cdots & v_{n+r-1}^{(1)} & v_{n+r-1}^{(0)} \\ \vdots & \cdots & \vdots & \vdots \\ v_n^{(r-1)} & \cdots & v_n^{(1)} & v_n^{(0)} \end{bmatrix}. \tag{3.3}$$

It is worth noting that Expressions (3.2)-(3.3) have been established first in [3], and they have been considered in the general setting in other papers such that [2, 10].

Let $\{J_n\}_{n\geq 0}$ be a given sequence of (r,p,k)-generalized Jacobsthal numbers, with initial conditions $\alpha_0,\ \dots,\ \alpha_{r-1}$. We can show that Expression (1.4) can be written under the matrix expression

$\mathbf{J}_{n+1} = \mathbf{B}\mathbf{J}_n,$ for every $n \geq r-1$, where $\mathbf{J}_n$ is the vector column $\mathbf{J}_n = (J_n,\ \ldots,\ J_{n-r+1})^t$ and $\mathbf{B}$ is the companion matrix,

$$\mathbf{B} = \begin{bmatrix} 1 & 2^p & 1 & \cdots & 1 & 2^k \\ 1 & 0 & & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.4}$$

An iterative process allows us to show that the matrix equation $\mathbf{J}_{n+1} = \mathbf{B}\mathbf{J}_n$ $(n \geq r-1)$, can be written under the form $\mathbf{J}_{n+r-1} = \mathbf{B}^n \mathbf{J}_{r-1}$, for every $n \geq 0$, where $\mathbf{J}_{r-1} = (\alpha_{r-1},\ \ldots,\ \alpha_0)^t$.

Theorem 2. *Under the preceding data, the entries $J_{ij}(n)$ of the powers $\mathbf{B}^n = (J_{ij}(n))_{0\leq i,j\leq r-1}$ are given by,*

$$J_{ij}(n) = J_{n+r-i-1}^{(r-j-1)} \tag{3.5}$$

where the sequences $\{J_n^{(s)}\}_{n\geq 0}$ $(0 \leq s \leq r-1)$ are defined by (2.1). In other words, for every $n \geq 0$, we have,

$$\mathbf{B}^n = \begin{bmatrix} J_{n+r-1}^{(r-1)} & \cdots & J_{n+r-1}^{(1)} & J_{n+r-1}^{(0)} \\ \vdots & \cdots & \vdots & \vdots \\ J_n^{(r-1)} & \cdots & J_n^{(1)} & J_n^{(0)} \end{bmatrix}. \tag{3.6}$$

Result of Theorem 2, namely, Expressions (3.5)-(3.6), will allow us to establish some identities related to the sequence of (r,p,k)-generalized Jacobsthal numbers.

Remark 2. *Note that the companion matrix* $\mathbf{A}$ *and its powers* $\mathbf{A}^n$ *have been considered for the matrix formulation of generalized Fibonacci sequences defined by a linear recurrence relation* $v_n = \sum_{i=0}^{r-1} a_i v_{n-i-1}$, *for* $n \geq r$, *with arbitrary initial conditions* $v_0,\ \ldots,\ v_{r-1}$, *in different papers in the literature (see, for instance, [4,6]). In [4] the entries of the powers* $\mathbf{A}^n$ *have been provided, using another family of elementary sequences similar to that defined by the expression* (3.1). *As shown in Theorem 1 of [3], the explicit formulas of the entries of the powers* $\mathbf{A}^n$ *have been established with the aid of the fundamental Fibonacci system* (3.1).

3.2 Jacobsthal Matrix formulation and its related identities

Let m, h in $\mathbf{N}$ and consider the two matrix powers $\mathbf{B}^m = (J_{ij}(m))_{0\leq i,\,j\leq r-1}$, $\mathbf{B}^h = (J_{ij}(h))_{0\leq i,\,j\leq r-1}$. Since the product of these powers of matrices is commutative, we have

$$\mathbf{B}^m\mathbf{B}^h = \mathbf{B}^h\mathbf{B}^m = \mathbf{B}^{m+h} = (J_{ij}(m+s))_{0\leq i,\,j\leq r-1}.$$

On the other side, we recall that for the product of two given square matrices $A = (a_{i,j})_{1\leq i,j\leq r}$ and $c = (b_{i,j})_{1\leq i,j\leq r}$, the entries of the product matrix $D = A.C = (d_{i,j})_{1\leq i,j\leq r}$ are given by $d_{i,j} = \sum_{k=1}^{r} a_{i,k}c_{k,j}$. Therefore, the entries $J_{ij}(m+h)$ of the matrix $\mathbf{B}^{m+h}$ are given as follows $J_{ij}(m+h) =$

$\sum_{k=0}^{r-1} J_{ik}(m)J_{kj}(h) = \sum_{k=0}^{r-1} J_{ik}(h)J_{kj}(m)$. Therefore, taking into account Expression (3.5) of the entries of the matrices powers $\mathbf{B}^m$ and $\mathbf{B}^s$, in terms of the fundamental system of sequences $\{J_n^{(s)}\}_{n\geq 0}$ $(0 \leq s \leq r-1)$ given by (2.1), we have the following important consequence of Theorem 2.

Proposition 3. *Let $\{J_n^{(s)}\}_{n\geq 0}$ $(0 \leq s \leq r-1)$ be the Jacobsthal fundamental Fibonacci system given by (2.1). Then, for every $m,\ h \geq 0$, we have*

$$J_{m+h+r-i-1}^{(r-j-1)} = \sum_{k=0}^{r-1} J_{m+r-i-1}^{(r-k-1)} J_{h+r-k-1}^{(r-j-1)} = \sum_{k=0}^{r-1} J_{h+r-i-1}^{(r-k-1)} J_{m+r-k-1}^{(r-j-1)}. \tag{3.7}$$

Let change the indexation in Expression (3.7), by setting $d = r-k-1$; $p = r-i-1$ and $q = r-j-1$. Then, we obtain the result.

Theorem 3. *Let $\{J_n^{(s)}\}_{n\geq 0}$ $(0 \leq s \leq r-1)$ be the Jacobsthal fundamental system (2.1). Then, any integers $m,\ h \geq 0$, we have*

$$J_{m+h+p}^{(q)} = \sum_{d=0}^{r-1} J_{m+p}^{(d)} J_{h+d}^{(q)} = \sum_{d=0}^{r-1} J_{h+p}^{(d)} J_{m+d}^{(q)}, \tag{3.8}$$

for every every p, q $(0 \leq p, q \leq r-1)$. Expression (3.8) *is called* (r,p,k)***-generalized Jacobsthal Identity***.

In addition, it was established in Lemmas 1, 2, that for every $s = 0,1,...,r-2$, the general term $J_n^{(s)}$ can be expressed only in term of $J_{n-p}^{(r-1)}$, as shown in Expressions (2.2), (2.3) and (2.4) namely, for every $s = 0,1,...,r-2$, we have the following the compact formula,

$$J_n^{(s)} = \sum_{j=0}^{s} a_{r-s+j-1} J_{n-j-1}^{(r-1)} = a_{r-s-1} J_{n-1}^{(r-1)} + a_{r-s} J_{n-2}^{(r-1)} + ... + a_{r-1} J_{n-s-1}^{(r-1)} \tag{3.9}$$

for every $n \geq r$, where $a_0 = a_2 = \cdots = a_{r-2} = 1$, $a_1 = 2^p$ and $a_{r-1} = 2^k$, for reason of convenience and simplicity. Expression (3.9) is still valid also for $i = r-1$, that is, in this case, we have $J_n^{(r-1)} = \sum_{j=0}^{r-1} a_{r-(r-1)+j-1} J_{n-j-1}^{(r-1)} = \sum_{j=0}^{r-1} a_j J_{n-j-1}^{(r-1)}$, which is nothing else but the recurrence relation (2.1), for the sequence $\{J_n^{(r-1)}\}_{n\geq 0}$. Moreover, for $s = 0$, we show easily that Expression (3.9) takes the form,

$$J_n^{(0)} = 2^k J_{n-1}^{(r-1)}, \text{ for every } n \geq r. \tag{3.10}$$

Now using Expression (3.9), we can establish that the identities (3.8) can be expressed only in terms of the fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$. That is, let first suppose that $q = r-1$ in (3.8). Then, for every $m \geq 0$ and $h \geq 0$, we have

$$J_{m+h+p}^{(r-1)} = \sum_{d=0}^{r-1} J_{m+p}^{(d)} J_{h+d}^{(r-1)} = \sum_{d=0}^{r-2} J_{m+p}^{(d)} J_{h+d}^{(r-1)} + J_{m+p}^{(r-1)} J_{h+r-1}^{(r-1)}. \tag{3.11}$$

Application of the formulas (2.2), (2.3) and (2.4) allows us to obtain $J^{(d)}_{m+p} = \sum_{j=0}^{d} a_{r-d+j-1} J^{(r-1)}_{m+p-j-1}$. Therefore, the substitution of this last expression of $J^{(d)}_{m+p}$ in the formula (3.11), allows us to derive $J^{(r-1)}_{m+h+p} = \sum_{d=0}^{r-2}\sum_{j=0}^{d} a_{r-d+j-1} J^{(r-1)}_{m+p-j-1} J^{(r-1)}_{h+d} + J^{(r-1)}_{m+p} J^{(r-1)}_{h+r-1}$. Second, suppose that $q=0$ in Expression (3.8), then, we have $J^{(0)}_{m+h+p} = \sum_{d=0}^{r-1} J^{(d)}_{m+p} J^{(0)}_{h+d} = \sum_{d=0}^{r-2} J^{(0)}_{h+d} \sum_{j=0}^{d} a_{r-d+j-1} J^{(r-1)}_{m+p-j-1}$. By considering Expression (3.10), we have $J^{(0)}_{n} = 2^k J^{(r-1)}_{n-1}$, for $n \geq r$. Hence, we get $2^k J^{(r-1)}_{m+h+p-1} = \sum_{d=0}^{r-1} 2^k J^{(r-1)}_{h+d-1} \sum_{j=0}^{d} a_{r-d+j-1} J^{(r-1)}_{m+p-j-1}$, for $n \geq r$. Since $a_{r-1} = 2^k \neq 0$ we derive that $J^{(r-1)}_{m+h+p-1} = \sum_{d=0}^{r-1}\sum_{j=0}^{d} a_{r-d+j-1} J^{(r-1)}_{m+p-j-1} J^{(r-1)}_{h+d-1}$, for $n \geq r$. Suppose that $1 \leq q \leq r-2$ in Expression (3.8). Thus, we obtain $J^{(q)}_{m+h+p} = J^{(r-1)}_{m+p} J^{(q)}_{h+d} + J^{(0)}_{m+p} J^{(q)}_{h+d} + \sum_{d=1}^{r-2} J^{(d)}_{m+p} J^{(q)}_{h+d}$, for $n \geq r$. Since $J^{(0)}_{n} = 2^k J^{(r-1)}_{n-1}$, for $n \geq r$, the former expression takes the form $J^{(q)}_{m+h+p} = (2^k+1) J^{(r-1)}_{m+p} J^{(q)}_{h+d} + \sum_{d=1}^{r-2} J^{(d)}_{m+p} J^{(q)}_{h+d}$. Now, let apply the formulas (2.2), (2.3) and (2.4) on both sides of the preceding expression, namely, $J^{(k)}_{n} = \sum_{j=0}^{k} a_{r-k+j-1} J^{(r-1)}_{n-j-1}$ $(0 \leq k \leq r-2)$, we obtain,

$$
\begin{aligned}
\sum_{j=0}^{q} a_{r-q+j-1} J^{(r-1)}_{m+h+p-j-1} &= (2^k+1) J^{(r-1)}_{m+p} + \sum_{j=0}^{q} a_{r-q+j-1} J^{(r-1)}_{h+d-j-1} + \sum_{d=1}^{r-2} J^{(d)}_{m+p} J^{(q)}_{h+d} \\
&= (2^k+1) J^{(r-1)}_{m+p} \sum_{j=0}^{q} a_{r-q+j-1} J^{(r-1)}_{h+d-j-1} + \Omega(r,m,h)
\end{aligned}
$$

for every $n \geq r$, where $\Omega(r,m,h) = \sum_{d=1}^{r-2}\sum_{k=0}^{d}\sum_{j=0}^{q} a_{r-q+k-1} a_{r-q+j-1} J^{(r-1)}_{m+p-k-1} J^{(r-1)}_{h+d-j-1}$. In summary, we can formulate the following result.

Theorem 4. *Let $\{J^{(r-1)}_{n}\}_{n\geq 0}$ be the Jacobsthal fundamental sequence and $m,h \geq 0$ two given integers. Then, using the identity (3.8), we get the following identities,*

$$
J^{(r-1)}_{m+h+p} = \sum_{d=0}^{r-2}\sum_{j=0}^{d} a_{r-d+j-1} J^{(r-1)}_{m+p-j-1} J^{(r-1)}_{h+d} + J^{(r-1)}_{m+p} J^{(r-1)}_{h+r-1}, \textit{ for } q = r-1, \tag{3.12}
$$

$$
J^{(r-1)}_{m+h+p-1} = \sum_{d=0}^{r-1}\sum_{j=0}^{d} a_{r-d+j-1} J^{(r-1)}_{m+p-j-1} J^{(r-1)}_{h+d-1}, \textit{ for } q = 0, \tag{3.13}
$$

and for $1 \leq q \leq r-2$ *we have*

$$\sum_{j=0}^{q} a_{r-q+j-1} J_{m+h+p-j-1}^{(r-1)} = (2^k+1) J_{m+p}^{(r-1)} \sum_{j=0}^{q} a_{r-q+j-1} J_{h+d-j-1}^{(r-1)} + \tag{3.14}$$
$$+ \sum_{d=1}^{r-2} \sum_{k=0}^{d} \sum_{j=0}^{q} a_{r-q+k-1} a_{r-q+j-1} \Delta(r,m,p,h,d,j,k),$$

where $a_0 = a_2 = \cdots = a_{r-2} = 1$, $a_1 = 2^p$ *and* $a_{r-1} = 2^k$, *and* $\Delta(r,m,p,h,d,j,k) = J_{m+p-k-1}^{(r-1)} J_{h+d-j-1}^{(r-1)}$.

The identities provided in Theorem 4 concerns the equality of the identity $J_{m+h+p}^{(q)} = \sum_{d=0}^{r-1} J_{m+p}^{(d)} J_{h+d}^{(q)}$. However, we can see that for the two members of the equality $\sum_{d=0}^{r-1} J_{m+p}^{(d)} J_{h+d}^{(q)} = \sum_{d=0}^{r-1} J_{h+p}^{(d)} J_{m+d}^{(q)}$, are symmetric by permuting the two integers m and s. Therefore, the permutation of the two integers m and s in Expressions (3.12), (3.13) and (3.14), will permit to get identical identities.

The linear identities (3.12), (3.13) and (3.14) have been established recently, for the Fibonacci-Pell numbers sequences, in the research papers [8, 9].

Remark 3. *The three identities* (3.12), (3.13) *and* (3.14) *can be formulated for the two special cases* $p = 0$ *and* $k \geq 1$ *or* $p \geq 1$ *and* $k = 0$. *That is, for* $p = 0$ *and* $k \geq 1$ *we have* $a_0 = a_1 = a_2 = \cdots = a_{r-1} = 1$ *and* $a_{r-1} = 2^k$ *in Expressions* (3.12)- (3.14), *and for* $p \geq 1$ *and* $k = 0$ *we have* $a_0 = a_2 = \cdots = a_{r-1} = 1$ *and* $a_1 = 2^p$ *in Expressions* (3.12)- (3.14).

4 COMBINATORIAL EXPRESSION OF THE (*r,p,k*)-GENERALIZED JACOBSTHAL NUMBERS AND RELATED IDENTITIES

4.1 Combinatorial aspect of (r,p,k)-generalized Jacobsthal numbers

Let $\{u_n\}_{n\geq 0}$ be the sequence defined by the combinatorial formula,

$$u_n = \sum_{j_0+2j_1+\cdots+rj_{r-1}=n-r+1} \frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\ldots j_{r-1}!} a_0^{j_0} a_1^{j_1} \ldots a_{r-1}^{j_{r-1}} \tag{4.1}$$

for every $n \geq r$, with $u_{r-1} = 1$, $u_j = 0$ for $j = 0,\ldots,r-2,$. It was established in [7] that the sequence $\{u_n\}_{n\geq 0}$ satisfies a linear recurrence relation of Fibonacci type. For $a_1 = 2^p$, $a_0 = a_2 = \cdots = a_{r-2} = 1$ and $a_{r-1} = 2^k$, we show that Expression (4.1) takes the following form,

$$w_n = \sum_{j_0+2j_1+\cdots+rj_{r-1}=n-r+1} \frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\ldots j_{r-1}!} 2^{j_1 p} 2^{j_{r-1} k} \tag{4.2}$$

where $w_{r-1}=1$, $w_j=0$ for $j=0,...,r-2$. In addition, the sequence $\{w_n\}_{n\geq 0}$ defined by (4.2), satisfies the generalized Jacobsthal recurrence relation (1.4), with initial conditions $w_{r-1}=1$, $w_j=0$ for $j=0,...,r-2$, namely, we have,

$$\begin{cases} w_{n+1}=w_n+2^p w_{n-1}+w_{n-2}+\cdots+w_{n-r+2}+2^k w_{n-r+1} \\ w_{r-1}=1,\ w_j=0 \text{ for } j=0,...,r-2. \end{cases}$$

Comparing the sequence $\{w_n\}_{n\geq 0}$ defined by (4.2) with the generalized Jacobsthal fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$, we show that these two sequences satisfy the linear recursive relation (1.4) and own the same initial conditions, namely, $J_{r-1}^{(r-1)}=w_{r-1}=1$, $J_n^{(r-1)}=w_n=0$ for $n=0,...,r-2$. Therefore, we get the following result.

Theorem 5. *Under the preceding data, the combinatorial expression of the Jacobsthal fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$, of the sequence of the (r,p,k)-generalized Jacobsthal numbers (1.4) is given by,*

$$J_n^{(r-1)}=\sum_{j_0+2j_1+\cdots+rj_{r-1}=n-r+1}\frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\ldots j_{r-1}!}2^{j_1p}2^{j_{r-1}k} \tag{4.3}$$

for every $n\geq r$, where $J_{r-1}^{(r-1)}=1$, $J_n^{(r-1)}=0$ for $n=0,...,r-2$.

For reason of convenience, we utilise the following usual combinatorial notation,

$$J_n^{(r-1)}=\rho(n+1,r)=\sum_{j_0+2j_1+\cdots+rj_{r-1}=n-r+1}\frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\ldots j_{r-1}!}2^{j_1p}2^{j_{r-1}k} \tag{4.4}$$

for every $n\geq r$, with $\rho(r,r)=1$ and $\rho(n,r)=0$ for $0\leq n\leq r-1$.

In the framework of the proof of Theorem 5, for establishing the combinatorial formula, we had used the procedure based on the equality of the Jacobsthal fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$ and the sequence $\{w_n\}_{n\geq 0}$ defined by Expression (4.2). However, generally in the literature to establish the combinatorial formula of the sequences defined by linear recurrence relations, one uses the associated generating function.

For the two special cases $p\geq 1$ and $k=0$ or $p=0$ and $k\geq 1$ of Expression (1.4), the combinatorial expression of Jacobsthal fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$ is formulated in the following corollary.

Corollary 3. *For $p=0$ the combinatorial expression of Jacobsthal fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$, namely (4.4), is given by*
$J_n^{(r-1)}=\sum_{j_0+2j_1+\cdots+rj_{r-1}=n-r+1}\frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\ldots j_{r-1}!}2^{j_{r-1}k}$ *and for $k=0$ it takes the form* $J_n^{(r-1)}=\sum_{j_0+2j_1+\cdots+rj_{r-1}=n-r+1}\frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\ldots j_{r-1}!}2^{j_1p}$*, for every $n\geq r$, where $J_{r-1}^{(r-1)}=1$, $J_n^{(r-1)}=0$ for $n=0,...,r-2$.*

A substitution process combining Proposition 2 and Theorem 5 allows us to derive the following result.

Proposition 4. *Let $\{J_n\}_{n\geq 0}$ be a sequence of the* $\mathbf{K}$*-vector space* $\mathscr{E}_{\mathbf{K}}^{(r)}(2^p;2^k)$*, of initial data* $\alpha_0,\cdots,\ \alpha_{r-1}$*. Then, for every $n\geq 0$, the combinatorial expression of the general term J_n is given under the form $J_n=\sum_{j=0}^{r-1} Z_j\rho(n-j,r)$, where $\rho(n,r)$* (4.4) *and the Z_j $(0\leq j\leq r-1)$ are given by* (2.7)*, namely,*

$$\begin{cases} Z_0=\alpha_0 2^k+\alpha_1+\cdots+\alpha_{r-3}+\alpha_{r-2}2^p+\alpha_{r-1} \\ Z_1=\alpha_1 2^k+\alpha_2+\cdots+\alpha_{r-2}+\alpha_{r-1}2^p \\ Z_j=\alpha_j 2^k+\alpha_{j+1}+\cdots+\alpha_{r-1},\ for\ 2\leq j\leq r-1. \end{cases}$$

Theorem 5 and Proposition 4, show that the combinatorial formula of the Jacobsthal fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$ and of every sequence of the (r,p,k)-generalized Jacobsthal numbers (1.4) are obtained without the use of their related generating functions.

The combinatorial expression of the (r,p,k)-generalized Jacobsthal fundamental sequence (1.4) for the two special cases $k=0$ and $p=0$, is formulated in the following corollary.

Corollary 4. *Under the data of Proposition 4, for $k=0$ the combinatorial expression of the general term J_n is given under the form $J_n=\sum_{j=0}^{r-1} Z_{k,j}\rho(n-j,r)$, where the $Z_{k,j}$ are given by* (2.8)*, namely,*

$$\begin{cases} Z_{p,0}=\alpha_0+\alpha_1+\cdots+\alpha_{r-3}+\alpha_{r-2}2^p+\alpha_{r-1} \\ Z_{p,1}=\alpha_1+\alpha_2+\cdots+\alpha_{r-2}+\alpha_{r-1}2^p \\ Z_{p,j}=\alpha_j+\alpha_{j+1}+\cdots+\alpha_{r-1},\ for\ 2\leq j\leq r-1. \end{cases}$$

And for $p=0$ we have $J_n=\sum_{j=0}^{r-1} Z_{k,j}\rho(n-j,r)$, where the $Z_{k,j}$ are given by (2.9)*, namely, $Z_{k,j}=\alpha_j 2^k+\sum_{d=j+1}^{r-1}\alpha_d$ for $0\leq j\leq r-1$.*

Remark 4. *When $p=1$ and $k=0$ and $p=0$ and $k=1$ (respectively) Corollaries 3 and 4 permit us to obtain the same properties for Expressions (1.2)- (1.3).*

4.2 Combinatorial identities related to (*r,p,k*)-generalized Jacobsthal numbers

In Subsection 3.2, we have established some identities related to the (r,p,k)-generalized Jacobsthal numbers (1.4), which are expressed in terms of the elements of the Jacobsthal fundamental Fibonacci system (2.1) and its related fundamental sequence. Especially, in Theorem 4 the identities (3.12), (3.13) and (3.14), are expressed in terms of Jacobsthal fundamental sequence $\{J_n^{(r-1)}\}_{n\geq 0}$. Then, combining results of Theorem 4 and Theorem 5, we can derive the analogous combinatorial identities. More precisely, substitution of the combinatorial Expression (4.4) in Expressions (3.12), (3.13) and (3.14) allows us to get the following result.

Theorem 6. *Let* $\{J_n^{(r-1)}\}_{n\geq 0}$ *be the Jacobsthal fundamental sequence and* m, $h \geq 0$ *are two given integers. Then, we have the following identities,*

$$\rho(m+p+h+1,r) = \sum_{d=0}^{r-2}\sum_{j=0}^{d} a_{r-d+j-1}\rho(m+p-j,r)\rho(h+d+1,r) + \rho(m+p+1,r)\rho(h+r,r) \quad (4.5)$$

for $q = r-1$*, and for* $q = 0$ *we have*

$$\rho(m+h+p+1,r) = \sum_{d=0}^{r-1}\sum_{j=0}^{d} a_{r-d+j-1}\rho(m+p-j,r)\rho(h+d,r) \quad (4.6)$$

finally, for $1 \leq q \leq r-2$ *we get*

$$\begin{aligned}\sum_{j=0}^{q} a_{r-q+j-1}\rho(m+h+p-j,r) &= (2^k+1)\rho(m+p,r)\sum_{j=0}^{q} a_{r-q+j-1}\rho(h+d-j,r) \quad (4.7)\\ &+ \sum_{d=1}^{r-2}\sum_{k=0}^{d}\sum_{j=0}^{q} a_{r-q+k-1}a_{r-q+j-1}\Delta(r,m,p,h,d,j,k),\end{aligned}$$

where $a_0 = a_2 = \cdots = a_{r-2} = 1$*,* $a_1 = 2^p$ *and* $a_{r-1} = 2^k$*, and* $\Delta(r,m,p,h,d,j,k) = \rho(m+p-k,r)\rho(h+d-j,r)$.

Remark 5. *Similarly to Remark 3, the three combinatorial identities* (4.5)*,* (4.6) *and* (4.7) *can be formulated for the special cases* $p = 0$*,* $k \geq 1$ *and* $p \geq 1$*,* $k = 0$.

Generally, in various articles of the literature the Chen-Louck's Theorem ([1, Theorem 3.1]) is used to express the combinatorial form of the entries of the powers of the companion matrix $\mathbf{A} = \mathbf{A}[a_0, a_1, \cdots, a_{r-1}]$. However, it was established in [7] that an explicit combinatorial formula a linear recurrence relation $v_n = \sum_{i=0}^{r-1} a_i v_{n-i-1}$, for $n \geq r$, with arbitrary initial conditions $v_0, \ldots, v_{r-1}$ is given in terms of $u_n = \rho(n,r)$ the the combinatorial formula 4.1. And with the aid of a direct computation using the identity $\dfrac{(k_0+\cdots+k_{r-1}-1)!}{k_0!\ldots(k_p-1)!\ldots k_{r-1}!} = \dfrac{k_p}{k_0+\cdots+k_{r-1}} \times \dfrac{(k_0+\cdots+k_{r-1})!}{k_0!\ldots k_{r-1}!}$ and Expressions (3.2), (4.1) it has been established in [10], that the Chen-Louck Theorem [1, Theorem 3.1] can be recovered by a direct calculation, as shown in [10, Proposition 3.1 (Chen–Louck's Theorem)].

5 CASSINI IDENTITY OF THE *(r,p,k)*-GENERALIZED JACOBSTHAL NUMBERS

In most works of literature, several identities like the Cassini identity, for some families of sequences of classical numbers and their generalization such that Fibonacci-Pell numbers or generalized Fibonacci-Pell numbers, are expressed in terms of the fundamental sequence (see [2,8,9]). Indeed, for the sequence of usual Fibonacci numbers and the sequence of usual Pell numbers, namely,

$$\begin{cases} F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1 \\ F_0 = \alpha_0, F_1 = \alpha_1 \end{cases} \qquad \begin{cases} P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 1 \\ P_0 = \alpha_0, P_1 = \alpha_1 \end{cases}$$

the Cassini identities are expressed in terms of their associated fundamental sequences, characterized by their initial conditions $F_0 = 0$, $F_1 = 1$ and $P_0 = 0$, $P_1 = 1$ (respectively). After expressing these Cassini identities of Fibonacci-Pell numbers under the determinant of the Casoratian matrix, associated to their fundamental Fibonacci systems, this approach has been extended to the generalized Fibonacci-Pell numbers in [2,8,9]. Let recall the well known formula of the determinant of a square matrix $M = (a_{ij})_{1\leq i,j\leq r}$, defined by $\det(M) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma)a_{\sigma(1),1}a_{\sigma(2),2}\ldots a_{\sigma(r),r}$, where $\mathscr{S}_r$ is the group of permutations σ of the r elements $\{1,2,\cdots,r\}$ and $sgn(\sigma)$ is the signature of the permutation σ. For the Jacobsthal matrix (3.4), its powers $\mathbf{B}^n = (J_{ij}^{(n)})_{0\leq i,j\leq r-1}$ are given by Expression (3.6). More precisely, the entries $J_{ij}(n)$ are expressed as in Expression (3.5), namely, $J_{ij}(n) = J_{n+r-i-1}^{(r-j-1)}$, where the sequences $\{J_n^{(s)}\}_{n\geq 0}$ $(0 \leq s \leq r-1)$ defined by (2.1), represent the Jacobsthal fundamental Fibonacci system. Therefore, we show that for the generalized Cassini identity for powers $\mathbf{B}^n = (J_{ij}^{(n)})_{0\leq i,j\leq r-1}$ takes the subsequent form,

$$C(n) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma)J_{\sigma(0),0}(n)J_{\sigma(1),1}(n)\cdots J_{\sigma(r-1),r-1}(n) = (-1)^{(r-1)n}2^{kn}, \tag{5.1}$$

where $\mathscr{S}_r$ is the group of permutations σ of the r elements $\{0,1,\cdots,r-1\}$. However, for the matrix powers $\mathbf{B}^n = (J_{ij}^{(n)})_{0\leq i,j\leq r-1}$, the entries $J_{ij}^{(n)}$ $(0 \leq i,j \leq r-1)$ are given by Expression (3.5). Therefore, using Expression (5.1), we come to have the result below.

Theorem 7. *Expression* (5.1) *of the generalized Cassini identity, related to the Jacobsthal fundamental Fibonacci system (2.1), is given under the following form,*

$$C(n) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma)J_{n+r-\sigma(0)-1}^{(r-1)}\ldots J_{n+r-\sigma(i)-1}^{(r-i-1)}\ldots J_{n+r-\sigma(r-1)-1}^{(0)} = (-1)^{(r-1)n}2^{kn}, \tag{5.2}$$

where $\mathscr{S}_r$ is the group of permutations of the set $\{0,1,\cdots,r-1\}$.

For the special case $k = 0$ of Expression (1.4) the Cassini is formulated as follows.

Corollary 5. *Under the data Theorem 7, for $k = 0$, the Cassini identity* (5.2) *takes the form $C(n) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma)J_{n+r-\sigma(0)-1}^{(r-1)}\cdots J_{n+r-\sigma(i)-1}^{(r-i-1)}\cdots J_{n+r-\sigma(r-1)-1}^{(0)} = (-1)^{(r-1)n}$, where $\mathscr{S}_r$ is the group of permutations σ of the set $\{0,1,\cdots,r-1\}$.*

For illustrate purpose of the preceding process, we consider below the special case $r = 3$.

Corollary 6. *For $r = 3$ is valid the generalized Cassini identity below,*

$$C(n) = \sum_{\sigma\in\mathscr{S}_3} sgn(\sigma)J_{n+2-\sigma(0)}^{(2)}J_{n+2-\sigma(1)}^{(1)}J_{n+2-\sigma(2)}^{(0)} = (-1)^{2n}2^{kn} = 2^{kn},$$

where $\mathscr{S}_3$ is the group of permutations of the set $\{0,1,3\}$.

On the other side, by setting $s=r-i-1$ and taking into account Expressions (2.3) and (2.4), we show that we have,

$$\begin{aligned} C(n) &= \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma) J^{(r-1)}_{n+r-\sigma(0)-1} J^{(r-2)}_{n+r-\sigma(1)-1} \cdots J^{(r-i-1)}_{n+r-\sigma(i)-1} \cdots J^{(0)}_{n+r-\sigma(r-1)-1} \\ &= \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma) J^{(r-1)}_{n+r-\sigma(0)-1} J^{(0)}_{n+r-\sigma(r-1)-1} \left[\prod_{s=2}^{r-3} J^{(s)}_{n+r-\sigma(r-s-1)-1} + J^{(r-2)}_{n+r-\sigma(1)-1} \right], \end{aligned}$$

since $\prod_{i=1}^{r-2} J^{(r-i-1)}_{n+r-\sigma(i)-1} = \prod_{s=1}^{r-2} J^{(s)}_{n+r-\sigma(r-s-1)-1} = \prod_{s=2}^{r-3} J^{(s)}_{n+r-\sigma(r-s-1)-1} + J^{(r-2)}_{n+r-\sigma(1)-1}$. Result of Lemmas 1 and 2, namely, Expressions (2.2), (2.3) and (2.4) shows that the Casoratian of the Jacobsthal fundamental Fibonacci system (2.1), can be also formulated in terms of the fundamental sequence $\{J^{(r-1)}_n\}_{n\geq 0}$. That is, we have

$$\begin{cases} J^{(0)}_{n+r-\sigma(r-1)-1} = 2^k J^{(r-1)}_{n+r-\sigma(r-1)-2}, \\ J^{(s)}_{n+r-\sigma(r-s-1)-1} = J^{(r-1)}_{n+r-\sigma(r-s-1)-2} + J^{(r-1)}_{n+r-\sigma(r-s-1)-3} + \cdots + J^{(r-1)}_{n+r-\sigma(r-s-1)-s-1} \\ \qquad + 2^k J^{(r-1)}_{n+r-\sigma(r-s-1)-s-2}, \\ J^{(r-2)}_{n+r-\sigma(1)-1} = 2^p J^{(r-1)}_{n+r-\sigma(1)-2} + J^{(r-1)}_{n+r-\sigma(1)-3} + \cdots + J^{(r-1)}_{+r-\sigma(1)-s-1} + 2^k J^{(r-1)}_{+r-\sigma(1)-s-2}. \end{cases}$$

Therefore, the substitution of the last expressions in Expression (5.2) allows us to obtain the Jacobsthal generalized Cassini in terms of the fundamental sequence, as shown in the next theorem.

Theorem 8. *Expression* (5.2) *of the generalized Jacobsthal Cassini identity, related to the Jacobsthal fundamental Fibonacci system (2.1), is stated under the form,*

$$C(n) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma) 2^k \Pi_{1,\sigma}(n) \Pi_{2,\sigma}(n) \prod_{s=1}^{r-3} \Pi_{s,d,\sigma}(n) = (-1)^{(r-1)n} 2^{kn}, \tag{5.3}$$

where

$$\begin{cases} \Pi_{1,\sigma}(n) = J^{(r-1)}_{n+r-\sigma(0)-1} J^{(r-1)}_{n+r-\sigma(r-1)-2}, \\ \Pi_{2,\sigma}(n) = 2^p J^{(r-1)}_{n+r-\sigma(1)-2} + \sum_{d=2}^{r-2} J^{(r-1)}_{n+r-\sigma(1)-d-1} + 2^k J^{(r-1)}_{n-\sigma(1)-2}, \\ \Pi_{s,d,\sigma}(n) = \sum_{d=1}^{s} J^{(r-1)}_{n+r-\sigma(r-s-1)-d-1} + 2^k J^{(r-1)}_{n+r-\sigma(r-s-1)-s-2}. \end{cases} \tag{5.4}$$

For the special cases $k=0$ and $p=0$ we have the following corollary, concerning the Cassini identity of the (r,p,k)-generalized Jacobsthal numbers (1.4).

Corollary 7. *Under the data of Theorem 8, for $k=0$ the Cassini identity* (5.3) *takes the form,*

$$C(n) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma) 2^k \Pi_{1,\sigma}(n) \Pi_{2,p,\sigma}(n) \prod_{s=1}^{r-3} \Pi_{s,d,\sigma}(n) = (-1)^{(r-1)n},$$

where

$$\begin{cases}\Pi_{1,\sigma}(n)=J^{(r-1)}_{n+r-\sigma(0)-1}J^{(r-1)}_{n+r-\sigma(r-1)-2},\\ \Pi_{2,p,\sigma}(n)=2^pJ^{(r-1)}_{n+r-\sigma(1)-2}+\sum_{d=2}^{r-2}J^{(r-1)}_{n+r-\sigma(1)-d-1}+J^{(r-1)}_{n-\sigma(1)-2},\\ \Pi_{s,d,\sigma}(n)=\sum_{d=1}^{s}J^{(r-1)}_{n+r-\sigma(r-s-1)-d-1}+2^kJ^{(r-1)}_{n+r-\sigma(r-s-1)-s-2},\end{cases}$$

and for $p=0$ *we have* $C(n)=\sum_{\sigma\in\mathscr{S}_r}sgn(\sigma)2^k\Pi_{1,\sigma}(n)\Pi_{2,k,\sigma}(n)\prod_{s=1}^{r-3}\Pi_{s,d,k,\sigma}(n)=(-1)^{(r-1)n}2^{kn}$,

where

$$\begin{cases}\Pi_{1,\sigma}(n)=J^{(r-1)}_{n+r-\sigma(0)-1}J^{(r-1)}_{n+r-\sigma(r-1)-2},\\ \Pi_{2,k,\sigma}(n)=J^{(r-1)}_{n+r-\sigma(1)-2}+\sum_{d=2}^{r-2}J^{(r-1)}_{n+r-\sigma(1)-d-1}+2^kJ^{(r-1)}_{n-\sigma(1)-2},\\ \Pi_{s,d,k,\sigma}(n)=\sum_{d=1}^{s}J^{(r-1)}_{n+r-\sigma(r-s-1)-d-1}+2^kJ^{(r-1)}_{n+r-\sigma(r-s-1)-s-2}.\end{cases}$$

Following results of Theorem 5, namely, Expression (4.3), and Expressions (5.3)-(5.4), we can derive the combinatorial expression of the Casoratian of the Jacobstal fundamental Fibonacci system. That is, with the aid o Expression (4.4), namely, $J_n^{(r-1)}=\rho(n+1,r)=\sum_{j_0+2j_1+\cdots+rj_{r-1}=n-r+1}\frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\dots j_{r-1}!}2^{j_1p}2^{j_{r-1}k}$, we derive that, for every s,d $(0\le s,d\le r-1)$, we have

$$J^{(r-1)}_{n+r-\sigma(r-s-1)-d-1}=\rho(n+r-\sigma(r-s-1)-d,r)=\sum_{\bigwedge(n,\sigma,s)}\frac{(j_0+\cdots+j_{r-1})!}{j_0!j_1!\dots j_{r-1}!}2^{j_1p}2^{j_{r-1}k},$$

where $\bigwedge(n,\sigma,s)=\{(j_0,j_1,\cdots,j_{r-1});\,j_0+2j_1+\cdots+rj_{r-1}=n+r-\sigma(r-s-1)-d-r\}$. Hence, by considering the two formulas (5.3)-(5.4) we have,

$$\begin{cases}J^{(r-1)}_{n+r-\sigma(0)-1}=\rho(n+r-\sigma(0),r),\\ J^{(r-1)}_{n+r-\sigma(r-s-1)-d-1}=\rho(n+r-\sigma(r-s-1)-d,r),\\ J^{(r-1)}_{n+r-\sigma(r-s-1)-s-2}=\rho(n+r-\sigma(r-s-1)-s-1,r).\end{cases}$$

Therefore, we can formulate the Jacobsthal combinatorial Cassini identity as follows.

Theorem 9. *Expression* (5.2) *of the* (r,p,k)*-generalized Jacobsthal Cassini identity, related to the Jacobsthal fundamental Fibonacci system (2.1), is stated under the form,*

$$C(n)=\sum_{\sigma\in\mathscr{S}_r}sgn(\sigma)2^k\Phi_{1,\sigma}(n)\Phi_{2,\sigma}(n)\prod_{s=1}^{r-3}\Phi_{s,d,\sigma}(n)=(-1)^{(r-1)n}2^{kn}\tag{5.5}$$

where $\Phi_{1,\sigma}(n)$, $\Phi_{2,\sigma}(n)$ *and* $\Phi_{s,d,\sigma}$ *are the following combinatorial expressions,*

$$\begin{cases}\Phi_{1,\sigma}(n)=\rho(n+r-\sigma(0),r)\rho(n+r-\sigma(r-1)-1,r),\\ \Phi_{2,\sigma}(n)=2^p\rho(n+r-\sigma(1)-1,r)+\sum_{d=2}^{r-2}\rho(n+r-\sigma(1)-d,r)+2^k\rho(n-\sigma(1)-1,r),\\ \Phi_{s,d,\sigma}(n)=\sum_{d=1}^{s}\rho(n+r-\sigma(r-s-1)-d,r)+2^k\rho(n+r-\sigma(r-s-1)-s-1,r).\end{cases}\tag{5.6}$$

Using Expressions (5.5)-(5.6), we get for the special cases $k=0$ and $p=0$ the following result concerning the combinatorial formula of the Cassini identity of the (r,p,k)-generalized Jacobsthal numbers (1.4).

Corollary 8. *Under the data of Theorem 9, for $k=0$ the Cassini identity (5.5) is given by* $C(n) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma)\Phi_{1,p,\sigma}(n)\Phi_{2,\sigma}(n)\prod_{s=1}^{r-3}\Phi_{s,d,\sigma}(n) = (-1)^{(r-1)n}$, *where*

$$\begin{cases} \Phi_{1,\sigma}(n) = \rho(n+r-\sigma(0),r)\rho(n+r-\sigma(r-1)-1,r), \\ \Phi_{2,p,\sigma}(n) = 2^p\rho(n+r-\sigma(1)-1,r)+\sum_{d=2}^{r-2}\rho(n+r-\sigma(1)-d,r)+\rho(n-\sigma(1)-1,r), \\ \Phi_{s,d,\sigma}(n) = \sum_{d=1}^{s}\rho(n+r-\sigma(r-s-1)-d,r)+\rho(n+r-\sigma(r-s-1)-s-1,r). \end{cases}$$

And for $p=0$, we have

$$C(n) = \sum_{\sigma\in\mathscr{S}_r} sgn(\sigma)2^k\Phi_{1,k,\sigma}(n)\Phi_{2,k,\sigma}(n)\prod_{s=1}^{r-3}\Phi_{s,d,k,\sigma}(n) = (-1)^{(r-1)n}2^{kn},$$

where

$$\begin{cases} \Phi_{1,\sigma}(n) = \rho(n+r-\sigma(0),r)\rho(n+r-\sigma(r-1)-1,r), \\ \Phi_{2,k,\sigma}(n) = \rho(n+r-\sigma(1)-1,r)+\sum_{d=2}^{r-2}\rho(n+r-\sigma(1)-d,r)+2^k\rho(n-\sigma(1)-1,r), \\ \Phi_{s,d,k,\sigma}(n) = \sum_{d=1}^{s}\rho(n+r-\sigma(r-s-1)-d,r)+2^k\rho(n+r-\sigma(r-s-1)-s-1,r). \end{cases}$$

It seems for us that the results of this section concerning the Cassini identities and its formulation in terms of the Jacobsthal fundamental sequence (5.3)-(5.4) and its related combinatorial identity (5.5)-(5.6) are not current in the literature.

Remark 6. *When $k=0$ and $p=1$ or $k=1$ and $p=0$ (respectively), Corollaries 5, 7 and 8 permit us to obtain the analogous properties for usual generalized Jacobsthal numbers (1.2)-(1.3).*

6 CONCLUDING REMARKS AND PERSPECTIVE

In this study we have presented new results regarding the model of (r,p,k)-generalized Jacobsthal numbers. Especially, we have based our construction on the Jacobsthal fundamental Fibonacci system and its related fundamental sequence. Therefore, the combinatorial expression of the (r,p,k)-generalized Jacobsthal numbers, as well as some related new identities are established. In addition the Cassini identities of the (r,p,k)-generalized Jacobsthal numbers are provided. Finally, some special cases are studied.

In the best of our knowledge our results are not current in the literature. Moreover our approaches can be applied for studying the analytical aspect of some classes of (r,p,k)-generalized Jacobsthal numbers defined by (1.4).

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