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Complementary Romanovski-Routh polynomials and their zeros

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ABSTRACT. The efficacy of numerical methods like integral estimates via Gaussian quadrature formulas depends on the localization of the zeros of the associated family of orthogonal polynomials. Following the renewed interest in quadrature formulas on the unit circle and R_{II} type polynomials, in this work we present properties satisfied by the zeros of the complementary Romanovski-Routh polynomials. Our results include extreme bounds, convexity, density, and the connection of such zeros with the zeros of classical orthogonal polynomials via asymptotic formulas.

Keywords: asymptotics, complementary Romanovski-Routh polynomials, convexity, density, extreme bounds.

1 INTRODUCTION

The n-th degree monic *complementary Romanovski-Routh polynomial* (CRR polynomial for short) that we consider here was presented in Martínez-Finkelshtein et al. [14] in terms of Gauss hypergeometric function as

$$\widehat{\mathscr{P}}_n(b;x) = (x-i)^n {}_2F_1\left(-n, b; b+\bar{b}; \frac{-2i}{x-i}\right), \quad n \geq 0,$$

where $b = \lambda + i\eta$, with $\lambda > 0$ and $\eta \in \mathbb{R}$. The CRR polynomials appear in wave functions in the context of quantum chromodynamics (see Raposo et al. [18]), and they also play an important role in the solution of the (one-dimensional) Schrödinger equation for the hyperbolic Scarf potential (see Gautschi [6]). We also cite Shukla and Swaminathan [19] for a recent generalization of these polynomials. It is noteworthy that the sequence of CRR polynomials is also an example of an Appel sequence. This property was explored recently in Martínez-Finkelshtein et al. [14] to

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obtain a generating function for these polynomials. The latter belongs to a subfamily of the Whittaker functions that includes the Coulomb wave functions and the Bessel functions as particular cases.

From the point of view of numerical analysis, in the recent work Bracciali et al. [1] it was shown that quadrature formulas based on these polynomials also provide useful quadrature formulas on the unit circle. Thus the study of the behavior and location of the zeros of the CRR polynomial is important.

We note that bounds for these zeros can be obtained from at least two distinct methods from the works of Jordaan and Toókos [12] and Martínez-Finkelshtein et al. [15]. In fact, in Jordaan and Toókos [12] the authors have investigated properties of the Pseudo-Jacobi polynomials, which can be adapted to the present case by exploring the connection between the CRR polynomials and the complexified Jacobi polynomials (Raposo et al. [18]). Alternatively, from the fact that CRR polynomials are related to special para-orthogonal polynomials (see Martínez-Finkelshtein et al. [14] for further details), the results found in Martínez-Finkelshtein et al. [15] can also be used to construct bounds for the zeros of CRR polynomials. However, the bounds obtained by these methods are not always optimal. In this work we shall present new bounds which are more accurate for large variations in the parameters λ and η. Furthermore, we also investigate properties such as convexity, density, and asymptotics. From the latter we also show how the CRR polynomial is linked to the classical Hermite and Laguerre polynomials.

In what follows, we adopt the more convenient normalization for the CRR polynomial $\mathscr{P}_n(b;x)$ (see Martínez-Finkelshtein et al. [14]) given by

$$\mathscr{P}_n(b;x) = \frac{(2\lambda)_n}{2^n(\lambda)_n}\widehat{\mathscr{P}_n}(b;x), \quad n \geq 0,$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ denotes the Pochhammer symbol, and $b = \lambda + i\eta$, $\lambda > 0$. In this work we study the zeros of $\mathscr{P}_n(b;x)$, henceforth denoted by $x_{n,k}^{(\eta,\lambda)}$, for $k = 1,2,\ldots,n$.

The paper is organized as follows. We leave to Section 2 the presentation of all the necessary notation, tools, and theorems which are going to be used throughout the work. We also recall in this section how the CRR polynomials are constructed via the R_{II}-type recurrence relation, and that their zeros coincide with the solutions of certain generalized eigenvalue problems. Sections 3 and 4 summarize the novel results of our work. Section 3 presents the extreme bounds, convexity, and density of the zeros of CRR polynomials. Moreover, these new bounds are compared with the two others obtained by the methods mentioned before, and it is shown through simulations that they are more accurate for several choices of parameters. We also state an orthogonality property of the k-th associated CRR polynomials. Finally, Section 4 is devoted to the asymptotic behavior of the CRR polynomials and their zeros.

2 PRELIMINARIES

The sequence $\{\mathscr{P}_n(b;x)\}_{n\geq 0}$ of CRR polynomials is solution of the R_{II}-type recurrence relation (see Ismail and Ranga [10])

$$\mathscr{P}_{n+1}(b;x) = (x - c_{n+1}^{(b)})\mathscr{P}_n(b;x) - d_{n+1}^{(\lambda)}(x^2+1)\mathscr{P}_{n-1}(b;x), \quad n \geq 1, \tag{2.1}$$

where $\mathscr{P}_0(b;x) = 1$, $\mathscr{P}_1(b;x) = x - c_1^{(b)}$,

$$c_n^{(b)} = \frac{\eta}{\lambda+n-1} \text{ and } d_{n+1}^{(\lambda)} = \frac{1}{4}\frac{n(n+2\lambda-1)}{(n+\lambda-1)(n+\lambda)}, \; n \geq 1. \tag{2.2}$$

The sequence $\{d_n^{(\lambda)}\}_{n\geq 2}$ is a positive chain sequence, whose detailed properties can be found for instance in Wall [23], and in Chihara [4]. Thus it follows (see Theorem 2.2 of Ismail and Ranga [10]) that the zeros of $\mathscr{P}_n(b;x)$ are real, simple, and can be denoted as

$$x_{n,1}^{(\eta,\lambda)} < x_{n,2}^{(\eta,\lambda)} < \ldots < x_{n,n-1}^{(\eta,\lambda)} < x_{n,n}^{(\eta,\lambda)}, \text{ for } n \geq 2.$$

It is also known that the zeros of $\mathscr{P}_{n-1}(b;x)$ and $\mathscr{P}_n(b;x)$ interlace.

Furthermore, $\mathscr{P}_n(b;x)$ is the characteristic polynomial of the generalized eigenvalue problem

$$\mathbf{A}_n^{(b)}\mathbf{u}_n^{(b)}(x) = x\mathbf{B}_n^{(b)}\mathbf{u}_n^{(b)}(x), \tag{2.3}$$

where the Hermitian tridiagonal matrices $\mathbf{A}_n^{(b)}$ and $\mathbf{B}_n^{(b)}$ are

$$\begin{bmatrix} c_1^{(b)} & i\sqrt{d_2^{(\lambda)}} & \ldots & 0 & 0 \\ -i\sqrt{d_2^{(\lambda)}} & c_2^{(b)} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & c_{n-1} & i\sqrt{d_n^{(\lambda)}} \\ 0 & 0 & \ldots & -i\sqrt{d_n^{(\lambda)}} & c_n^{(b)} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & \sqrt{d_2^{(\lambda)}} & \ldots & 0 & 0 \\ \sqrt{d_2^{(\lambda)}} & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \sqrt{d_n^{(\lambda)}} \\ 0 & 0 & \ldots & \sqrt{d_n^{(\lambda)}} & 1 \end{bmatrix},$$

respectively. Polynomial sequences generated by R_{II} type recurrence relations were initially investigated by Ismail and Mason [9], and their link to generalized eigenvalue problems was first explored by Zhedanov [24]. Another property presented by the n-th degree CRR polynomial used throughout the work is that it satisfies the second order differential equation

$$A(x)\mathscr{P}_n''(b;x) - 2B(n,\lambda,\eta;x)\mathscr{P}_n'(b;x) + C(n,\lambda)\mathscr{P}_n(b;x) = 0, \tag{2.4}$$

with $A(x) = x^2+1$, $B(n,\lambda,\eta;x) = (n+\lambda-1)x-\eta$ and $C(n,\lambda) = n(n+2\lambda-1)$. This equation is used in conjunction with the following results.

Theorem 2.1 (Sturm Comparison Theorem). *Let y and Y be nontrivial solutions of the differential equations*

$$\begin{aligned} y''(x)+f(x)y(x) &= 0, \\ Y''(x)+F(x)Y(x) &= 0, \end{aligned}$$

respectively, where $f, F \in C(r,s)$ are continuous functions in the interval (r,s), and $f(x) \le F(x)$, $f \not\equiv F$, in (r,s). Let x_1 and x_2 with $r < x_1 < x_2 < s$ be two consecutive zeros of y. Then the function Y has at least one sign variation in the interval (x_1,x_2). The Sturm Comparison Theorem can be found in Szegő's [22] or in Hille's book [7]. We use it to obtain the Theorem 3.4, and the Corollary 3.5 describing the density of the zeros of CRR polynomials and their k-associated CRR polynomials defined in (3.4). We recall that the Sturm Comparison Theorem provides bounds for the zeros of several special functions (see Szegő [22]). Moreover, it can be applied to prove the Convexity Theorem, which is convenient to study the convexity and spacing between consecutive zeros of polynomial functions. For further reading about convexity of some classical orthogonal polynomials we cite Jordaan and Toókos [11]. We use the Convexity Theorem with some appropriated variable transformation to demonstrate the Theorem 3.2.

Theorem 2.2 (Convexity Theorem). *Let y be a nontrivial solution of differential equation*

$$y''(x)+f(x)y(x) = 0,$$

where $f \in C(r,s)$ is a continuous function in the interval (r,s) and $x_1 < \ldots < x_n$ are the zeros of y, also in the interval (r,s).

- **I**. *If f is decreasing in (r,s), then $x_k - x_{k-1} < x_{k+1} - x_k$ for each $k = 2,\ldots,n-1$;*
- **II**. *If f is increasing in (r,s), then $x_k - x_{k-1} > x_{k+1} - x_k$ for each $k = 2,\ldots,n-1$.*

*The zeros of y are called convex (Resp. concave) on the interval (r,s) if **I** (Resp. **II**) holds.*

Finally, in order to present upper and lower bounds for the largest and smallest zero of the CRR polynomial, we need the next technical lemma from Dimitrov and Genno [5], which is a particular case of an inequality that holds for functions in the Laguerre-Pólya class. Further information on the latter can be found, for instance, in Nikolov and Uluchevin [17].

Lemma 2.3. *Let f be a polynomial with only real zeros and degree $n \ge 4$. If $f(\zeta) = 0$, then*

$$3(n-2)[f''(\zeta)]^2 - 4(n-1)f'(\zeta)f'''(\zeta) \ge 0.$$

3 EXTREME BOUNDS, CONVEXITY AND DENSITY

We start this section by presenting the bounds for the largest and the smallest zero of the polynomial $\mathscr{P}_n(b;x)$ in the next theorem, which is applied later on to obtain inequalities for the distance between two consecutive zeros of CRR polynomial in the Corollary 3.3.

Theorem 3.1. *Let $\lambda > 0$ and $n \geq 4$. Then*

$$\frac{\eta[n^2+2\lambda+n(\lambda-1)]-(n-1)\sqrt{\Delta_n}}{\lambda[2(n-1)+(n+2)\lambda]} < x_{n,1}^{(\eta,\lambda)}$$

$$x_{n,n}^{(\eta,\lambda)} < \frac{\eta[n^2+2\lambda+n(\lambda-1)]+(n-1)\sqrt{\Delta_n}}{\lambda[2(n-1)+(n+2)\lambda]}$$

where

$$\Delta_n = (\eta n)^2+(\lambda n)^2+2\lambda n(n-3)+2\lambda n(\eta^2+\lambda^2+2\lambda)+4\lambda[\eta^2+(\lambda-1)^2].$$

Proof. We know that $\mathscr{P}_n(b;x)$ satisfies Equation (2.4). Let $y(x) = \mathscr{P}_n(b;x)$. If $y(\zeta) = \mathscr{P}_n(b;\zeta) = 0$, then straightforward manipulations lead to

$$y''(\zeta) = 2\frac{(n+\lambda-1)\zeta-\eta}{\zeta^2+1}y'(\zeta), \tag{3.1}$$

and

$$y'''(\zeta) = \frac{2[(n+\lambda-2)\zeta-\eta]y''(\zeta)-[(n-2)(n+\lambda-1)+n\lambda]y'(\zeta)}{\zeta^2+1}.$$

Thus, from Lemma 2.3, we have

$$\left[2\frac{y'(\zeta)}{\zeta^2+1}\right]^2 Q(\zeta) \geq 0,$$

where we defined $Q(x) = q_2x^2+q_1x+q_0$ with

$$\begin{aligned}
q_2 &= -\lambda\left[(n+2)\lambda+2(n-1)\right],\\
q_1 &= 2\eta\left[n(n+\lambda-1)+2\lambda\right],\\
q_0 &= n^3+2(\lambda-2)n^2-\left(\eta^2+4\lambda-5\right)n-2\left(\eta^2-\lambda+1\right).
\end{aligned}$$

After some manipulations, we obtain that the discriminat is $4(n-1)^2\Delta_n$, where Δ_n is defined in the theorem's statement. Since $\lambda > 0$ and $n \geq 4$, each term of Δ_n is either nonnegative or strictly positive. Therefor, the two real roots of the polynomial $Q(x)$ are

$$x_{1,2} = \frac{\eta[n(n+\lambda-1)+2\lambda]\pm(n-1)\sqrt{\Delta_n}}{\lambda[(n+2)\lambda+2(n-1)]}.$$

Since q_2 is negative, it follows that all the zeros of $\mathscr{P}_n(b;x)$ are in the interval (x_1,x_2). □

As anticipated, we solved numerically for the largest and smallest zeros of some representative CRR polynomials, and the corresponding bounds build from Theorem 3.1. Moreover, those bounds were tested and compared with previous bounds found in Jordaan and Toókos [12] and Martínez-Finkelshtein et al. [15], which can be determined with a few preliminary manipulations. The bounds found in Jordaan and Toókos [12] are connected to the CRR polynomials via their relation with the complexifield Jacobi polynomials, whereas the bounds given in Martínez-Finkelshtein et al. [15] follow from the connection between $\mathscr{P}_n(b;x)$ and the para-orthogonal polynomials explored in Section 2 of Martínez-Finkelshtein et al. [14].

Our findings are summarized in Tables 1 and 2. The columns labeled by RRY contain the results from Theorem 3.1, whereas the ones labeled by JT and MRV contain the results found with aid of Jordaan and Toókos [12] and Martínez-Finkelshtein et al. [15], respectively. Moreover, for a given set of parameters, sharper bounds were highlighted, and the lower and upper bounds are discriminated by the subscripts *min* and *max*, respectively. Our simulations for different values of n, λ and η reveal some interesting patterns. As shown in Table 1, by fixing $\eta = 2$ and $n = 30$, the bounds given by Theorem 3.1 are less accurate than the previous ones when the parameter λ is small. For example, for $\lambda = 0.75$ and $\lambda = 1.75$, the bounds obtained in Jordaan and Toókos [12] and Martínez-Finkelshtein et al. [15] are sharper. However, as λ increases the bounds from Theorem 3.1 are always sharper, as shown in Table 1 for λ as $5, 10, 15, 20, 25$ and 70. Furthermore, even for smaller values of λ, Theorem 3.1 seems to always provide better bounds if η is not too small, as shown in Table 2.

Table 1: Smallest and largest zeros, and extreme bounds when $\eta = 2$ and $n = 30$. The RRY_{max} and RRY_{min} are the upper and lower extreme bounds, respectively, for the zeros obtained from Theorem 3.1. JT_{max}, MRV_{max}, JT_{min} and MRV_{min} are the upper and lower extreme bounds obtained by Jordaan and Toókos [12] and Martínez-Finkelshtein et al. [15], respectively.

λ	$x_{n,1}^{(\eta,\lambda)}$	RRY_{min}	MRV_{min}	JT_{min}
0.75	−3.60366	−6.23369	−6.40957	**−4.97174**
1.75	−3.56614	−5.49264	**−5.12489**	−5.13691
5	−2.94731	**−3.83491**	−4.12943	−4.05698
10	−2.24406	**−2.69331**	−2.96588	−2.87276
15	−1.84833	**−2.14342**	−2.34765	−2.28402
20	−1.59757	**−1.81694**	−1.97708	−1.93278
25	−1.42313	**−1.59827**	−1.72983	−1.69740
70	−0.83027	**−0.89890**	−0.95410	−0.94747
λ	$x_{n,n}^{(\eta,\lambda)}$	RRY_{max}	MRV_{max}	JT_{max}
0.75	51.93148	64.38003	133.35633	**62.30239**
1.75	18.22559	24.05906	**19.87006**	24.48929
5	5.99823	**7.61473**	8.41346	8.05698
10	3.31253	**3.95257**	4.44314	4.20609
15	2.43428	**2.81256**	3.11345	2.98990
20	1.98455	**2.24960**	2.46186	2.38732
25	1.70595	**1.90970**	2.07325	2.02333
70	0.90430	**0.97623**	1.03484	1.02684

Table 2: Smallest and largest zeros, and extreme bounds when $\lambda = 1.5$ and $n = 4$. The RRY_{max} and RRY_{min} are the upper and lower extreme bounds, respectively, for the zeros obtained from Theorem 3.1. JT_{max}, MRV_{max}, JT_{min} and MRV_{min} are the upper and lower extreme bounds obtained by Jordaan and Toókos [12] and Martínez-Finkelshtein et al. [15], respectively.

η	$x_{n,1}^{(\eta,\lambda)}$	RRY_{min}	MRV_{min}	JT_{min}
0	−1.18804	−1.41421	**−1.28644**	−1.61803
0.5	−0.81060	−1.00000	**−0.97680**	−1.13951
1.5	−0.31265	−0.43303	−0.50471	**−0.41745**
5	−1.84833	**−2.14342**	−2.34765	−2.28402
7	0.96538	**0.91036**	0.76104	0.85254
15	2.33259	**2.25266**	1.98870	2.07881
35	5.58634	**5.41883**	4.83529	4.99376
90	14.43584	**14.01430**	12.52994	12.91264
η	$x_{n,n}^{(\eta,\lambda)}$	RRY_{max}	MRV_{max}	JT_{max}
0	1.18804	1.41421	**1.28644**	1.61803
0.5	1.68062	1.9333	**1.69224**	2.18714
1.5	2.94807	3.23303	**3.31012**	3.56030
5	2.43428	**2.81256**	3.11345	2.98990
7	11.54147	**12.15630**	13.75984	12.96786
15	24.51550	**25.74734**	29.32890	27.39236
35	57.08690	**59.91450**	68.35048	63.70115
90	146.73775	**153.98571**	175.71725	163.69722

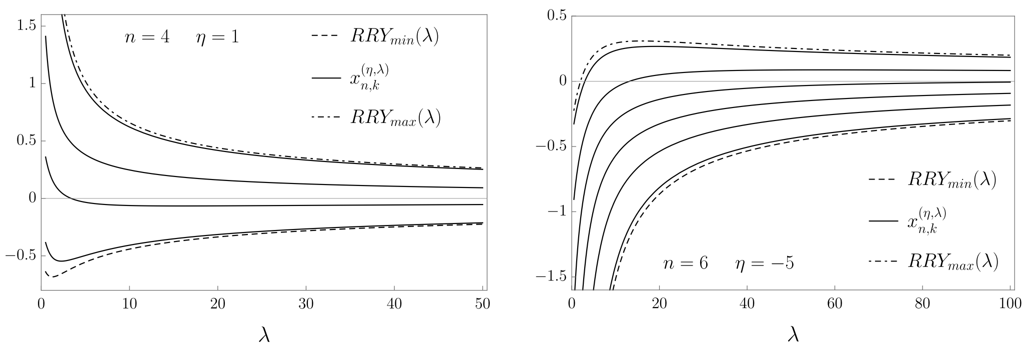

Figure 1: The inner curves represent the graphs of $x_{n,k}^{(\eta,\lambda)}$, $k = 1,\ldots,n$, as functions of λ for fixed values of n and η. The RRY_{min} and RRY_{max} are the upper and lower extreme bounds, respectively, for the zeros obtained from Theorem 3.1.

Numerical simulations for different values of λ also show that for larger values of λ the results given by Theorem 3.1 are somewhat optimal. In fact, in Figure 1 we have plotted the zeros and the bounds as functions of the parameter λ for two different and fixed sets of values n and η. In Figure 1 the continuous curves from bottom ($k=1$) to top ($k=n$) represent the plots of $x_{n,k}^{(\eta,\lambda)}$ as a function of λ with a fixed n and η. The lower dashed curve represents the lower extreme bound for the zeros and the upper dot-dashed curve represents the upper extreme bound for the zeros, both obtained from Theorem 3.1. Clearly, these simulations suggest that for larger values of λ the upper dot-dashed curve approaches the curve of $x_{n,n}^{(\eta,\lambda)}$ and similarly for the lower dashed curve with the one from $x_{n,1}^{(\eta,\lambda)}$.

The following theorem gives some information about the convexity of the zeros of the n-th CRR polynomial with respect to the parameters λ and η.

Theorem 3.2. *Let $M_n = M_n(b) = \eta(n+\lambda)/\lambda(\lambda-1)$ with $b = \lambda + i\eta$, $\lambda > 0$ and $\eta \in \mathbb{R}$. If $\lambda > 1$ and*

I. $\eta = 0$, *then the negative (Resp. positive) zeros of $\mathscr{P}_n(b;x)$ are concave (Resp. convex);*

II. $\eta > 0$, *then the negative zeros of $\mathscr{P}_n(b;x)$ are concave and the positive zeros of $\mathscr{P}_n(b;x+M_n)$ are convex;*

III. $\eta < 0$, *then the negative zeros of $\mathscr{P}_n(b;x+M_n)$ are concave and the positive zeros of $\mathscr{P}_n(b;x)$ are convex.*

If $0 < \lambda < 1$ and

VI. $\eta > 0$, *then the zeros of $\mathscr{P}_n(b;x)$ in $(M_n, 0)$ are concave;*

V. $\eta < 0$, *then the zeros of $\mathscr{P}_n(b;x)$ in $(0, M_n)$ are convex;*

If $\lambda = 1$ and $\eta > 0$ (Resp. $\eta < 0$), then all negative (Resp. positive) zeros are concave (Resp. convex).

Proof. The Differential Equation (2.4) in the canonical form form reads

$$v_n''(\theta) + \Lambda_n(\theta) v_n(\theta) = 0, \tag{3.2}$$

where

$$\begin{aligned} \Lambda_n(\theta) &= \lambda(1-\lambda)\tan^2(\theta) + 2\eta(n+\lambda)\tan(\theta) + n^2 + 2n\lambda + \lambda - \eta^2, \\ v_n(\theta) &= e^{\eta\theta}\cos(\theta)^{n+\lambda}\, \mathscr{P}_n(b;\tan(\theta)), \end{aligned} \tag{3.3}$$

with $\theta \in (-\pi/2, \pi/2)$. The details of the transformation to the Sturm-Liouville form can be found in Section 1.8 of Szegő [22]. We note that the zeros $\theta_{n,k}^{(\eta,\lambda)}$ of $v_n(\theta)$ are such that $x_{n,k}^{(\eta,\lambda)} = \tan(\theta_{n,k}^{(\eta,\lambda)})$ and

$$-\pi/2 < \theta_{n,1}^{(\eta,\lambda)} < \theta_{n,2}^{(\eta,\lambda)} < \ldots < \theta_{n,n}^{(\eta,\lambda)} < \pi/2.$$

From Equation (3.3) it follows that

$$\frac{d\Lambda_n(\theta)}{d\theta} = \frac{2}{\cos^2(\theta)}[\eta(n+\lambda) - \lambda(\lambda-1)\tan(\theta)] = 0$$

if and only if $\tan(\theta) = \eta(n+\lambda)/\lambda(\lambda-1)$.

Suppose first that $\lambda > 1$. Then if $\eta > 0$, we have that $M_n > 0$, Λ_n is decreasing in the interval $(\arctan(M_n), \pi/2)$, and increasing in $(-\pi/2, \arctan(M_n))$. Therefore, by Theorem 2.2, the negative zeros of $v_n(\theta)$ are concave and the positive zeros of $v_n(\theta)$ that belong to the interval $(\arctan(M_n), \pi/2)$ are convex. Since the tangent function is concave in $(-\pi/2, 0)$, convex in $(\arctan(M_n), \pi/2) \subset (0, \pi/2)$ and increasing in $(-\pi/2, \pi/2)$, we conclude that the negative zeros of $\mathscr{P}_n(b;x)$ are concave and the positive zeros of $\mathscr{P}_n(b;x)$ which are in the interval (M_n, ∞) are convex. When $\eta < 0$, the statement is a consequence of the identity $(-1)^n \mathscr{P}_n(\overline{b}; -x) = \mathscr{P}_n(b;x)$, where $\overline{b} = \lambda - i\eta$, and for the case of $\eta = 0$, the proof is immediate.

Finally, the proof when $0 < \lambda < 1$ is analogous, and for the remaining $\lambda = 1$ case, we note that sign$(d\Lambda_n(\theta)/d\theta)$=sign$(\eta)$, where $\mathrm{sign}(\eta) = 1$ if $\eta \geq 0$ and $\mathrm{sign}(\eta) = -1$ if $\eta < 0$, and the conclusion follows immediately by Theorem 2.2. □

It should be remarked that our result does not establish the convexity behavior for all the zeros for a given parameter choice. For instance, when $\lambda > 1$ and $\eta > 0$, the above theorem asserts that the zeros located on the right hand side of M_n are convex and the ones located on the left hand side of the origin are concave. However, for this case, our result cannot be used to draw any conclusion about the zeros located in the interval $(0, M_n)$.

The next corollary presents inequalities for the parametrized distance between two consecutive zeros of the n-th CRR polynomial.

Corollary 3.3. *Let* $-\pi/2 < \theta_{n,1}^{(\eta,\lambda)} < \theta_{n,2}^{(\eta,\lambda)} < \ldots < \theta_{n,n}^{(\eta,\lambda)} < \pi/2$ *be given by* $\theta_{n,k}^{(\eta,\lambda)} = \arctan(x_{n,k}^{(\eta,\lambda)})$*, for* $k = 1, \ldots, n$*, and let*

$$f_n^{(\eta,\lambda)} = [n^2 + \lambda(2n+1)] \left\{ 1 - \frac{\eta^2}{\lambda(1-\lambda)} \right\},$$
$$g_n^{(\eta)} = (n+1)^2 + \frac{1}{3n} \left\{ \eta^2[n(n+1)^2 + n^3 + 4] + \mathrm{sign}(\eta)\, 2\eta(n^2-1)\sqrt{\eta^2(n+1)^2 + 3(\eta^2+n^2)} \right\},$$

where $\mathrm{sign}(\eta) = 1$ *if* $\eta \geq 0$ *and* $\mathrm{sign}(\eta) = -1$ *if* $\eta < 0$.

I. *If* $0 < \lambda < 1$ *and* $\eta^2 \leq \lambda(1-\lambda)$*, then*

$$\theta_{n,k+1}^{(\eta,\lambda)} - \theta_{n,k}^{(\eta,\lambda)} \leq \left(\frac{\pi^2}{f_n^{(\eta,\lambda)}} \right)^{1/2}, \quad k = 1, \ldots, n-1.$$

II. *If $\lambda > 1$ and $\eta \in \mathbb{R}$, then*

$$\theta_{n,k+1}^{(\eta,\lambda)} - \theta_{n,k}^{(\eta,\lambda)} \geq \left(\frac{\pi^2}{f_n^{(\eta,\lambda)}} \right)^{1/2}, \quad k = 1, \dots, n-1.$$

III. *If $\lambda = 1$, $\eta \in \mathbb{R}$ and $n \geq 4$, then*

$$\theta_{n,k+1}^{(\eta,\lambda)} - \theta_{n,k}^{(\eta,\lambda)} > \left(\frac{\pi^2}{g_n^{(\eta)}} \right)^{1/2}, \quad k = 1, \dots, n-1.$$

Proof. Notice that $\Delta_n(\theta)$, given in (3.3), has a single critical point for $\lambda \neq 1$ at $\theta = \arctan(M_n)$, where M_n was defined in Theorem 3.2. Under the condition **I**, we have $\Delta_n(\theta) \geq f_n^{(\eta,\lambda)}$, and since $y(\theta) = \sin[(f_n^{(\eta,\lambda)})^{1/2}(\theta - \theta_{n,k}^{(\eta,\lambda)})]$ is the solution of the differential equation $y''(\theta) + f_n^{(\eta,\lambda)} y(\theta) = 0$, the result is a consequence of Theorem 2.1. The condition **II** follows analogously. In view of **III**, we have $\Delta_n(\theta) = 2\eta(n+1)\tan(\theta) + (n+1)^2 - \eta^2$. By Theorem 3.1 the zeros of $\mathscr{P}_n(b;x)$, for $n \geq 4$, are in the interval (x_1, x_2) where

$$x_{1,2} = \frac{\eta(n^2+2) \pm (n-1)\sqrt{\eta^2(n^2+2n+4)+3n^2}}{3n}.$$

Therefore, $\arctan(x_1) \leq \theta \leq \arctan(x_2)$ implies that $\Delta_n(\theta) < 2\eta(n+1)x_2 + (n+1)^2 - \eta^2$ for $\eta \geq 0$ and $\Delta_n(\theta) < 2\eta(n+1)x_1 + (n+1)^2 - \eta^2$ for $\eta < 0$, and thus $\Delta_n(\theta) < g_n^{(\eta)}$. $\square$

The next theorem was inspired by the construction in Moak et al. [16] used to obtain information about the density of the zeros of Jacobi polynomials.

Theorem 3.4. *Let $\mathscr{X}^{(b)}$ be the set of all the zeros of $\mathscr{P}_n(b;x)$ for every $n \geq 1$. If $\lambda \geq 1$, then $\mathscr{X}^{(b)}$ is dense in $\mathbb{R}$.*

Proof. Let Θ_{t_1} and Θ_{t_2} be arbitrary numbers in $(-\pi/2, \pi/2)$ such that

$$\tan(\Theta_{t_1}) = t_1, \;\; \tan(\Theta_{t_2}) = t_2 \;\text{ and }\; \Theta_M = \max(|\Theta_{t_1}|, |\Theta_{t_2}|).$$

We first show that for a given $\varepsilon > 0$, there exists n_1 such that all consecutive zeros $\theta_{n_1,k}^{(\eta,\lambda)}$ and $\theta_{n_1,k+1}^{(\eta,\lambda)}$ of $v_{n_1}(\theta)$ within $(\Theta_{t_1}, \Theta_{t_2})$ are such that $\theta_{n_1,k+1}^{(\eta,\lambda)} - \theta_{n_1,k}^{(\eta,\lambda)} < \varepsilon$. If $\lambda > 1$ and $(n^2 + 2n\lambda + \lambda - n - \eta^2) > 0$, consider the two roots $\Theta_1^{(n)}$ and $\Theta_2^{(n)}$ of $\Lambda_n - n$ given by

$$\tan(\Theta_{1,2}^{(n)}) = \frac{\eta(n+\lambda) \pm \sqrt{\eta^2(n+\lambda)^2 + \lambda(\lambda-1)(n^2+2n\lambda+\lambda-n-\eta^2)}}{\lambda(\lambda-1)},$$

where $\tan(\Theta_2^{(n)}) > \tan(\Theta_1^{(n)})$ and $\Lambda_n(\theta)$ is defined in (3.3). Since $\lambda > 1$, we have $\Lambda_n(\theta) - n > 0$ for $\theta \in (\Theta_1^{(n)}, \Theta_2^{(n)})$. Without much difficulty, we can also verify that $\Theta_1^{(n)}$ tends to $-\pi/2$ and $\Theta_2^{(n)}$ tends to $\pi/2$ when n tends to ∞. Let n_0 be such that for all $n \geq n_0$, $\Theta_1^{(n)} < \Theta_{t_1}$ and $\Theta_2^{(n)} > \Theta_{t_2}$. Now we note that the roots of $\sin(\sqrt{n}\theta)$, which is the solution of the differential equation $Y'' + nY = 0$, are dense in the real line as n tends to ∞. Therefore, for a given $\varepsilon > 0$, let n_1 be such that $n_1 > n_0$ and $2\pi/\sqrt{n_1} < \varepsilon$. By Theorem 2.1 it follows that $v_{n_1}(\theta)$ has a zero between any two consecutive zeros of $\sin(\sqrt{n_1}\theta)$ in $(\Theta_{t_1}, \Theta_{t_2})$. Thus, we conclude that there are infinitely many zeros of $v_n(\theta)$ in $(\Theta_{t_1}, \Theta_{t_2})$ as n tends to ∞, and that if $\theta_{n_1,k}^{(\eta,\lambda)}$ and $\theta_{n_1,k+1}^{(\eta,\lambda)}$ are in $(\Theta_{t_1}, \Theta_{t_2})$, then $\theta_{n_1,k+1}^{(\eta,\lambda)} - \theta_{n_1,k}^{(\eta,\lambda)} < \varepsilon$. Hence, if $x_{n_1,k+1}^{(\eta,\lambda)}$ and $x_{n_1,k}^{(\eta,\lambda)}$ are in (t_1, t_2), then from $\tan(A-B) = [\tan(A) - \tan(B)]/[1 + \tan(A)\tan(B)]$,

$$\begin{aligned} x_{n_1,k+1}^{(\eta,\lambda)} - x_{n_1,k}^{(\eta,\lambda)} &= \left[1 + \tan(\theta_{n_1,k+1}^{(\eta,\lambda)})\tan(\theta_{n_1,k}^{(\eta,\lambda)})\right]\tan(\theta_{n_1,k+1}^{(\eta,\lambda)} - \theta_{n_1,k}^{(\eta,\lambda)}) \\ &< \left[1 + \tan^2(\Theta_M)\right]\tan(\varepsilon). \end{aligned}$$

Now let $\lambda = 1$. Then the equation $\Lambda_n(\theta) - n = 0$ has the root

$$\tan(\Theta_1^{(n)}) = -\frac{n^2 + n + 1 - \eta^2}{2\eta(n+1)}.$$

Note that when $\eta > 0$, it follows that $\Lambda_n(\theta) - n > 0$ for $\theta \in (\Theta_1^{(n)}, \pi/2)$ and that $\Theta_1^{(n)}$ tends to $-\pi/2$ as n tends to ∞. By using the same argument as before, we conclude that $\mathscr{X}^{(b)}$ is dense in $\mathbb{R}$ for $\eta > 0$. The case $\eta < 0$ follows from the identity $(-1)^n \mathscr{P}_n(\bar{b}; -x) = \mathscr{P}_n(b;x)$, with $b = \lambda + i\eta$. For $\eta = 0$, the solution of Equation (3.2) is a linear combination of $\cos((n+1)\theta)$ and $\sin((n+1)\theta)$, whose zeros are dense in $\mathbb{R}$ as n tends to ∞. This concludes the proof of the theorem. $\square$

We now consider the polynomials $\{\mathscr{P}_n^{(k)}(b;x)\}_{n\geq 0}$, the k-th associated CRR polynomials, obtained from

$$\mathscr{P}_{n+1}^{(k)}(b;x) = (x - c_{k+n+1}^{(b)})\mathscr{P}_n^{(k)}(b;x) - d_{k+n+1}^{(\lambda)}(x^2+1)\mathscr{P}_{n-1}^{(k)}(b;x), \quad n \geq 1, \tag{3.4}$$

where $\mathscr{P}_0^{(k)}(b;x) = 1$, $\mathscr{P}_1^{(k)}(b;x) = x - c_{k+1}^{(b)}$. The coefficients $c_n^{(b)}$ and $d_{n+1}^{(\lambda)}$ are as in (2.2). Clearly, $\mathscr{P}_n^{(0)}(b;x) = \mathscr{P}_n(b;x)$, $n \geq 1$.

Then Theorem 3.4 has the corollary

Corollary 3.5. *For any fixed nonnegative integer k, let $\mathscr{X}_k^{(b)}$ be the set of all the zeros of $\mathscr{P}_n^{(k)}(b;x)$, for every $n \geq 1$. If $\lambda \geq 1$, then $\mathscr{X}_k^{(b)}$ is dense in $\mathbb{R}$.*

Proof. Consider the positive chain sequence $\{d_n^{(\lambda)}\}_{n\geq 2}$ (see (3.5) and (3.6) below for more details). It follows that for any $k \geq 1$ the sequence $\{d_{k+n}^{(\lambda)}\}_{n\geq 2}$ is also a positive chain sequence (details in Chihara [4]). Consequently, in the same way as shown in Ismail and Ranga [10], it follows that the zeros of $\mathscr{P}_n^{(k)}(b;x)$ are all real, simple and also that the zeros of $\mathscr{P}_{n-1}^{(k)}(b;x)$ and $\mathscr{P}_n^{(k)}(b;x)$ interlace.

From the three term recurrence in Equation (3.4) it is also straightforward to show that

$$\mathscr{P}_n^{(k+1)}(b;x)\mathscr{P}_n^{(k)}(b;x) - \mathscr{P}_{n-1}^{(k+1)}(b;x)\mathscr{P}_{n+1}^{(k)}(b;x) = (1+x^2)^n \prod_{j=2}^{n+1} d_{k+j}^{(\lambda)} \; > \; 0,$$

for all real x and $n \geq 1$. Thus, the zeros of $\mathscr{P}_n^{(k+1)}(b;x)$ and $\mathscr{P}_{n+1}^{(k)}(b;x)$ also interlace.

Hence, the corollary follows via induction. $\square$

We now provide some information regarding the orthogonality properties of the sequence of polynomials $\{\mathscr{P}_n^{(k)}(b;x)\}_{n\geq 0}$. These orthogonality properties are very much related to what it is known about the positive chain sequence $\{d_{k+n}^{(\lambda)}\}_{n\geq 2}$, and the Verblunsky coefficients associated with orthogonal polynomials on the unit circle. For more information about Verblunsky coefficients and orthogonal polynomials on the unit circle we cite, for example, Ismail [8] and Simon [20].

The sequence $\{d_n^{(\lambda)}\}_{n\geq 2}$ is one of the nicest example of a positive chain sequence, where one can state its minimal $\{m_n^{(\lambda)}\}_{n\geq 1}$ and maximal $\{M_n^{(\lambda)}\}_{n\geq 1} = \{M_n^{(\lambda;0)}\}_{n\geq 1}$ parameter sequences explicitly as follows:

$$m_n^{(\lambda)} = M_n^{(\lambda;0)} = \frac{1}{2}\frac{n-1}{\lambda+n-1}, \quad n \geq 1, \tag{3.5}$$

if $0 < \lambda \leq 1/2$ and

$$m_n^{(\lambda)} = \frac{1}{2}\frac{n-1}{\lambda+n-1} \quad \text{and} \quad M_n^{(\lambda;0)} = \frac{1}{2}\frac{2\lambda+n-2}{\lambda+n-1}, \quad n \geq 1, \tag{3.6}$$

if $\lambda > 1/2$.

Hence, for any $k \geq 1$, there follows (see Chihara [4], page 94) that the maximal $\{M_n^{(\lambda;k)}\}_{n\geq 1}$ parameter sequences of the positive chain sequence $\{d_{k+n}^{(\lambda)}\}_{n\geq 2}$ satisfy

$$M_n^{(\lambda;k)} = \frac{1}{2}\frac{k+n-1}{\lambda+k+n-1}, \quad n \geq 1, \quad \text{if} \quad 0 < \lambda \leq 1/2,$$

$$M_n^{(\lambda;k)} = \frac{1}{2}\frac{2\lambda+k+n-2}{\lambda+k+n-1}, \quad n \geq 1, \quad \text{if} \quad \lambda > 1/2.$$

Thus, we can state the following:

Theorem 3.6. *Assume that either $k = 0$ and $\lambda > 1/2$ or $k \geq 1$ and $\lambda > 0$. Let $\nu^{(b;k)}$ be the probability measure on the unit circle such that its Verblunsky coefficients are*

$$\beta_{n-1}^{(b;k)} = \frac{1}{\tau_{n-1}^{(b;k)}}\frac{1-M_n^{(\lambda;k)} - ic_{k+n}^{(b)}}{1-ic_{k+n}^{(b)}}, \quad n \geq 1,$$

where $\tau_0^{(b;k)} = 1$ and $\tau_n^{(b;k)} = \tau_{n-1}^{(b;k)}(1-ic_{k+n}^{(b)})/(1+ic_{k+n}^{(b)})$.
If $d\varphi^{(b;k)}(x) = d\nu^{(b;k)}((x+i)/(x-i))$ then

$$\int_{-\infty}^{\infty} x^j \frac{\mathscr{P}_n^{(k)}(b;x)}{(x^2+1)^n} d\varphi^{(b;k)}(x) = \gamma_n^{(b;k)}\,\delta_{n,j}, \quad j = 0,1,\ldots,n,$$

where $\gamma_0^{(b;k)} = 1$ and $\gamma_n^{(b;k)} = (1-M_n^{(\lambda;k)})\gamma_{n-1}^{(b;k)}$, $n \geq 1$. Moreover,

$$\mathscr{P}_n^{(k+1)}(b;x) = \int_{-\infty}^{\infty} \frac{(1+x^2)^n \mathscr{P}_n^{(k)}(t) - (1+t^2)^n \mathscr{P}_n^{(k)}(x)}{M_1^{(\lambda;k)}(t-x)(1+t^2)^n} d\varphi^{(b;k)}(t). \tag{3.7}$$

Proof. The orthogonality relation follows from Theorem 1.2 of Ismail and Ranga [10]. To obtain Formula (3.7) we use results found in Section 3 of Bracciali et al. [1]. □

In general, it is not possible to write down the exact expression for $\varphi^{(b;k)}$, with the exception

$$d\varphi^{(b;0)}(x) = \frac{2^{2\lambda-1}|\Gamma(b)|^2}{\Gamma(2\lambda-1)} \frac{e^{\eta\pi}}{2\pi} \frac{(e^{-\cot^{-1}x})^{2\eta}}{(1+x^2)^\lambda} dx.$$

4 ASYMPTOTICS

Continuing our proposed program, we now study the asymptotics of CRR polynomial. Recall that the these polynomials are parametrized by $b = \lambda + i\eta$. We shall focus here on the asymptotic properties of $\mathscr{P}_n(b;x)$ with respect to both parameters and their link to the Hermite and Laguerre polynomials. In order to show it, we first recall that the polynomials $H_n(x)$ and $L_n^{(\alpha)}(x)$ are solutions of the recurrence formulas

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 1,$$

with $H_0(x) = 1$, $H_1(x) = 2x$, and

$$L_{n+1}^{(\alpha)}(x) = \left(2 + \frac{\alpha - 1 - x}{n+1}\right) L_n^{(\alpha)}(x) - \left(1 + \frac{\alpha - 1}{n+1}\right) L_{n-1}^{(\alpha)}(x), \quad n \geq 1,$$

with $L_0^{(\alpha)}(x) = 1$, $L_1^{(\alpha)}(x) = \alpha + 1 - x$, respectively. We also recall that $\{H_n(x)\}_{n\geq 0}$ is a sequence of orthogonal polynomials in $(-\infty,\infty)$ with respect to the Gaussian weight e^{-x^2} and $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ is orthogonal in $(0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$.

Theorem 4.1. *For $b = \lambda + i\eta$ and each $n \geq 0$, let*

$$\widehat{U}_n(b;x) = \frac{2^n(\lambda)_n}{\lambda^{n/2}} \mathscr{P}_n(b;x/\sqrt{\lambda}) \quad \text{and} \quad \widetilde{U}_n(b;x) = \frac{(\lambda)_n}{\eta^n n!} x^n \mathscr{P}_n(b;2\eta/x).$$

Then the following formulas hold:

$$\begin{aligned} \widehat{U}_n(b;x) &= H_n(x) - 2\eta n H_{n-1}(x)\frac{1}{\sqrt{\lambda}} + \mathscr{R}_1\big((1/\sqrt{\lambda})^2\big), \\ \widetilde{U}_n(b;x) &= L_n^{(2\lambda-1)}(x) + \mathscr{R}_2\big(1/\eta^2\big), \end{aligned}$$

where $\mathscr{R}_i(h^k)$, for $i = 1,2$, denote a sum of terms proportional to h^j with $j \geq k$.

Proof. From Equation (2.1), we first observe that the sequence of functions $\{\widehat{U}_n(b;x)\}_{n\geq 0}$ satisfies

$$\begin{aligned} &\widehat{U}_0(b;x) = H_0(x) = 1, \quad \widehat{U}_1(b;x) = 2x - \tfrac{2\eta}{\sqrt{\lambda}} \quad \text{and} \\ &\widehat{U}_{n+1}(b;x) = 2[x - \alpha_{n+1}(x)]\widehat{U}_n(b;x) - [2n + \beta_{n+1}(x)]\widehat{U}_{n-1}(b;x), \end{aligned} \tag{4.1}$$

for $n \geq 1$, where

$$\alpha_{n+1}(x) = \frac{\eta}{\sqrt{\lambda}} - \frac{n}{\lambda}x \ \text{ and } \ \beta_{n+1}(x) = \frac{n[2x^2 + n - 1]}{\lambda} + \frac{n(n-1)x^2}{\lambda^2}.$$

Hence, $\widehat{U}_1(b;x) = H_1(x) - \frac{2\eta}{\sqrt{\lambda}} H_0(x)$ and the claim follows trivially for $\widehat{U}_1(b;x)$. Suppose that

$$\widehat{U}_k(b;x) = H_k(x) - \frac{2\eta k}{\sqrt{\lambda}} H_{k-1}(x) + \mathscr{R}_1\big((1/\sqrt{\lambda})^2\big)$$

for each $k = 1,\ldots,n$. By the Recurrence (4.1), we have

$$\begin{aligned}
\widehat{U}_{n+1}(b;x) &= 2\left(x - \frac{\eta}{\sqrt{\lambda}}\right)\widehat{U}_n(b;x) - 2n\widehat{U}_{n-1}(b;x) + \mathscr{R}_1\big((1/\sqrt{\lambda})^2\big) \\
&= 2\left(x - \frac{\eta}{\sqrt{\lambda}}\right)\left[H_n(x) - \frac{2\eta n}{\sqrt{\lambda}} H_{n-1}(x)\right] - 2n\left[H_{n-1}(x) - \frac{2\eta(n-1)}{\sqrt{\lambda}} H_{n-2}(x)\right] \\
&\quad + \mathscr{R}_1\big((1/\sqrt{\lambda})^2\big) \\
&= 2xH_n(x) - 2nH_{n-1}(x) - \frac{2\eta}{\sqrt{\lambda}}\left[H_n(x) + 2nxH_{n-1}(x) - 2n(n-1)H_{n-2}(x)\right] \\
&\quad + \mathscr{R}_1\big((1/\sqrt{\lambda})^2\big).
\end{aligned}$$

Thus,

$$\widehat{U}_{n+1}(b;x) = H_{n+1}(x) - \frac{2\eta(n+1)}{\sqrt{\lambda}} H_n(x) + \mathscr{R}_1\big((1/\sqrt{\lambda})^2\big),$$

and this proves the expansion formula for $\widehat{U}_n(b;x)$, $n \geq 2$.

Now to obtain the second expansion formula, we first observe that

$$\widetilde{U}_0(b;x) = 1 = L_0^{(2\lambda-1)}(x), \quad \widetilde{U}_1(b;x) = 2\lambda - x = L_1^{(2\lambda-1)}(x),$$

and

$$\widetilde{U}_{n+1}(b;x) = \left[2 + \frac{2\lambda - 2 - x}{n+1}\right]\widetilde{U}_n(b;x) - \left[1 + \frac{2\lambda - 2}{n+1}\right]\left[1 + \left(\frac{x}{2\eta}\right)^2\right]\widetilde{U}_{n-1}(b;x),$$

for $n \geq 1$. Thus, the formula is obtained similarly. □

We note that the asymptotic behavior with respect to the parameter η can also be deduced from the connection between the Romanovski-Routh polynomials and the Bessel polynomials, as shown in Lesky et al. [13]. The next theorem presents the order of convergence for the zeros of CRR polynomial as functions of λ and η.

Theorem 4.2. *Let $h_{n,k}$ and $l_{n,k}^{(2\lambda-1)}$ be the k-th zeros (in increasing order) of $H_n(x)$ and $L_n^{(2\lambda-1)}(x)$, respectively. Then*

$$x_{n,k}^{(\eta,\lambda)} = \frac{h_{n,k}}{\sqrt{\lambda}} + \frac{\eta}{\lambda} + \mathscr{O}\big((1/\sqrt{\lambda})^3\big) \quad \text{and} \quad x_{n,k}^{(\eta,\lambda)} = \frac{2\eta}{l_{n,n+1-k}^{(2\lambda-1)}} + \mathscr{O}(1/\eta),$$

for each $k = 1,\ldots,n$, as $\lambda \to \infty$ and $\eta \to \infty$, respectively.

Proof. Observe from Theorem 4.1 that $\widehat{U}_n(b;x) \to H_n(x)$ as $\lambda \to \infty$. Since the zeros of $\widehat{U}_n(b;x)$ are given by $\sqrt{\lambda}\, x_{n,k}^{(\lambda,\eta)}$, it follows that $\sqrt{\lambda}\, x_{n,k}^{(\lambda,\eta)} \to h_{n,k}$ as $\lambda \to \infty$, where $h_{n,k}$ denote the zeros of the Hermite polynomial of degree n. This asymptotic behavior implies that the λ dependence of $x_{n,k}^{(\eta,\lambda)}$ can be modeled by the ansatz [3]

$$x_{n,k}^{(\eta,\lambda)} = \frac{h_{n,k}}{\sqrt{\lambda}} + \frac{B_{n,k}}{\lambda} + \mathscr{O}\big((1/\sqrt{\lambda})^3\big). \tag{4.2}$$

Then our claim holds true if we show that $B_{n,k} = \eta$ for $k = 1,2,\ldots,n$.

By straightforward calculations we can show that

$$\frac{\mathscr{P}_n''(b; x_{n,k}^{(\eta,\lambda)})}{2\mathscr{P}_n'(b; x_{n,k}^{(\eta,\lambda)})} = \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{x_{n,k}^{(\eta,\lambda)} - x_{n,i}^{(\eta,\lambda)}}.$$

Thus, from Equation (3.1) we have

$$\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{x_{n,k}^{(\eta,\lambda)} - x_{n,i}^{(\eta,\lambda)}} = \frac{(n-1+\lambda)x_{n,k}^{(\eta,\lambda)} - \eta}{1 + (x_{n,k}^{(\eta,\lambda)})^2}, \quad \text{for} \quad k = 1,2,\ldots n. \tag{4.3}$$

Moreover, by using Equation (4.2) in the left hand side of (4.3) and then considering the formal series expansion in powers of $1/\sqrt{\lambda}$, we have

$$\begin{aligned}
\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{x_{n,k}^{(\eta,\lambda)} - x_{n,i}^{(\eta,\lambda)}} &= \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{\sqrt{\lambda}}{h_{n,k} - h_{n,i} + (B_{n,k} - B_{n,i})/\sqrt{\lambda} + \mathscr{O}\big((1/\sqrt{\lambda})^2\big)} \\
&= \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{\sqrt{\lambda}}{h_{n,k} - h_{n,i}} - \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{B_{n,k} - B_{n,i}}{(h_{n,k} - h_{n,i})^2} + \mathscr{O}\big(1/\sqrt{\lambda}\big),
\end{aligned}$$

for $k = 1,2,\ldots,n$. Similarly, substitution of Equation (4.2) in the right hand side of (4.3) gives

$$\begin{aligned}
\frac{(n-1+\lambda)x_{n,k}^{(\eta,\lambda)} - \eta}{1 + (x_{n,k}^{(\eta,\lambda)})^2} &= \frac{(n-1+\lambda)\left[h_{n,k}/\sqrt{\lambda} + B_{n,k}/\lambda + \mathscr{O}\big((1/\sqrt{\lambda})^3\big)\right] - \eta}{1 + h_{n,k}^2/\lambda + \mathscr{O}\big((1/\sqrt{\lambda})^3\big)} \\
&= \sqrt{\lambda}h_{n,k} + B_{n,k} - \eta + \mathscr{O}\big(1/\sqrt{\lambda}\big),
\end{aligned}$$

for $k = 1,2,\ldots,n$. Thus, by comparing the above equalities, we find that the constraints

$$\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{h_{n,k} - h_{n,i}} = h_{n,k} \quad \text{and} \quad -\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{B_{n,k} - B_{n,i}}{(h_{n,k} - h_{n,i})^2} = B_{n,k} - \eta, \tag{4.4}$$

should be satisfied for each $k = 1,\ldots,n$. The first one involving the zeros of Hermite polynomials is known to be true. It is commonly known as Stieltjes system of equations for the zeros of the Hermite polynomials, studied by Stieltjes in 1885 (see, for example, Stieltjes [21] and Szegő [22]).

The second constraint in (4.4) is a $n \times n$ linear system of equations that can be put in the form $\mathbf{Tv} = -\mathbf{v}$, where the matrix $\mathbf{T} = (t_{k,j})_{k,j=1}^{n}$ is symmetric with diagonal and off-diagonal elements given, respectively, by

$$t_{k,k} = \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{(h_{n,k} - h_{n,i})^2} \quad \text{and} \quad t_{k,j} = -\frac{1}{(h_{n,j} - h_{n,k})^2}.$$

Moreover, $\mathbf{v} = [v_1, v_2, \ldots, v_n]^t$ with $v_k = B_{n,k} - \eta$. The matrix $\mathbf{T}$ is positive semi-definite (for instance, in Calogero [2] its eigenvalues are explictly given) and thus its eigenvalues are nonnegative. This implies that the unique solution of the system $\mathbf{Tv} = -\mathbf{v}$ is the trivial solution. That is, $v_k = B_{n,k} - \eta = 0$ for $k = 1, \ldots, n$. This proves the asymptotics with respect to the parameter λ.

To establish the asymptotic behavior with respect to the parameter η, suppose that

$$x_{n,k}^{(\eta,\lambda)} = 2\eta D_{n,k} + C_{n,k} + \mathscr{O}(1/\eta). \tag{4.5}$$

From Theorem 4.1 we must have $D_{n,k} = 1/l_{n,n+1-k}^{(2\lambda-1)}$. Substitution of Equation (4.5) in Equation (4.3), followed by the expansion in powers of $1/\eta$ leads to the constraints

$$\begin{aligned} \sum_{\substack{i=1\\i\neq k}}^{n} \frac{1}{D_{n,k} - D_{n,i}} &= \frac{2(n+\lambda-1)D_{n,k} - 1}{2D_{n,k}^2}, \\ \sum_{\substack{i=1\\i\neq k}}^{n} \frac{C_{n,k} - C_{n,i}}{(D_{n,k} - D_{n,i})^2} &= \frac{C_{n,k}[(n+\lambda-1)D_{n,k} - 1]}{D_{n,k}^3}, \end{aligned} \tag{4.6}$$

for $k = 1, 2, \ldots, n$. The first expression involving the reciprocal zeros of Laguerre polynomials is the nonlinear system satisfied by the zeros of the special Bessel polynomial, namely, $x^n L_n^{(2\lambda-1)}(1/x)$.

In terms of $l_{n,n+1-k}^{(2\lambda-1)}$, Equations (4.6) are equivalent to

$$\begin{aligned} \sum_{\substack{i=1\\i\neq k}}^{n} \frac{l_{n,n+1-i}^{(2\lambda-1)}}{l_{n,n+1-i}^{(2\lambda-1)} - l_{n,n+1-k}^{(2\lambda-1)}} &= (n+\lambda-1) - \frac{1}{2} l_{n,n+1-k}^{(2\lambda-1)}, \\ \sum_{\substack{i=1\\i\neq k}}^{n} \frac{[l_{n,n+1-i}^{(2\lambda-1)}]^2 (C_{n,k} - C_{n,i})}{\left(l_{n,n+1-i}^{(2\lambda-1)} - l_{n,n+1-k}^{(2\lambda-1)}\right)^2} &= C_{n,k}\left[(n+\lambda-1) - l_{n,n+1-k}^{(2\lambda-1)}\right], \end{aligned} \tag{4.7}$$

for $k = 1, 2, \ldots, n$.

Now substituting for $(n+\lambda-1) - l_{n,n+1-k}^{(2\lambda-1)}/2$ in the right hand side of the second equation in (4.7) with the first equation, we find the system

$$\sum_{\substack{i=1\\i\neq k}}^{n} \left[\frac{l_{n,n+1-i}^{(2\lambda-1)} l_{n,n+1-k}^{(2\lambda-1)} C_{n,k}}{\left(l_{n,n+1-i}^{(2\lambda-1)} - l_{n,n+1-k}^{(2\lambda-1)}\right)^2} - \frac{l_{n,n+1-i}^{(2\lambda-1)} l_{n,n+1-i}^{(2\lambda-1)} C_{n,i}}{\left(l_{n,n+1-i}^{(2\lambda-1)} - l_{n,n+1-k}^{(2\lambda-1)}\right)^2} \right] = -\frac{1}{2} l_{n,n+1-k}^{(2\lambda-1)} C_{n,k},$$

for $k = 1, 2, \ldots, n$. We now consider this as a system of equations written as

$$\tilde{\mathbf{T}}\mathbf{u} = -\frac{1}{2}\mathbf{u}, \quad \mathbf{u} = [l_{n,n}^{(2\lambda-1)} C_{n,1}, l_{n,n-1}^{(2\lambda-1)} C_{n,2}, \ldots, l_{n,1}^{(2\lambda-1)} C_{n,n}]^T.$$

The matrix $\tilde{\mathbf{T}}$ is positive semi-definite, which follows from the Gershgorin Theorem. This system was also considered in Calogero [2] where its eigenvalues were calculated. Thus, with the negative factor $-1/2$, the system $\tilde{\mathbf{T}}\mathbf{u} = -\frac{1}{2}\mathbf{u}$ has only the trivial solution. That is, $l_{n,n+1-k}^{(2\lambda-1)} C_{n,k} = 0$, $k = 1, 2, \ldots, n$ and hence, $C_{n,k} = 0$, $k = 1, 2, \ldots, n$. □

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Data availability

All data generated or analysed during this study are included in this published article.

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