

On Variational Calculus With Fuzzy Initial Condition and Free Final Condition

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ABSTRACT. Many mathematical problems involve some kind of subjectivity. This paper studies the problem of variational calculus with fuzzy initial condition and free final condition. The solution is obtained by applying the Zadeh's extension to the classical solution, with respect to initial condition. Using a partial order relation of for fuzzy numbers, a theorem that ensuring the optimality of this solution is proved.

Keywords: Zadeh's extension, fuzzy, variational calculus.

1 INTRODUCTION

The calculation of variations constitutes a generalization of the elementary calculation of maxima and minima of real functions of one variable. While, the of variations deals with optimization of functional in function spaces, while ordinary calculus deals with real variables. Functional can, for example, be formed by integrals involving an unknown function and its derivatives. The interest is in extreme functions - those that make the functional reach a maximum or minimum value.

The variational problem with free final condition is commonly stated as

$$\min_{x \in C^1([t_0, t_f])} J(x) = \int_{t_0}^{t_f} F(t, x, \dot{x}) dt \quad (1.1)$$
$$x(0) = x_0 \text{ and } x(t_f) \text{ is free,}$$

where F is a real function, x_0 is a real number and $\mathcal{C}^1[t_0, t_f]$ is the space of continuously differentiable functions from the closed interval $[t_0, t_f]$ to $\mathbb{R}$.

Problems in variational calculus, as well as optimization problems in real spaces, have been investigated within fuzzy environments, although the literature in this area is still relatively limited.

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In [9], the authors introduce a version of the necessary optimality conditions (Euler–Lagrange equation) for fuzzy variational problems using the Buckley–Feuring derivatives. In [8], an Euler–Lagrange equation is also obtained, this time based on the concept of α -differentiability. In [7], the generalized Hukuhara derivative is employed to derive Euler–Lagrange conditions for both unconstrained and isoperimetric problems. Furthermore, using Caputo-type fractional derivatives, [13] presents a variant of the Euler–Lagrange equation for fuzzy fractional variational calculus, relying on the strongly generalized Hukuhara derivative.

In this article, we address variational problems subject to a fuzzy initial condition and a free final condition. Based on the idea developed in [5], which studies the variational problem with a fuzzy initial condition but with a fixed terminal condition, our goal is not to establish necessary or sufficient optimality conditions through any specific notion of fuzzy differentiability. Rather, we demonstrate that, under a prescribed partial order on fuzzy numbers, the classical solution of the variational problem generates, via Zadeh’s extension principle, the corresponding solution in the fuzzy setting. In this sense, we show that the Zadeh extension with respect to the classical initial condition fully characterizes the solution of the fuzzy variational problem with free terminal condition.

2 PRELIMINARIES

The concepts and results described here on fuzzy set theory, as well as more details, can be found in [1, 3, 11, 15].

Definition 1. (*Fuzzy set*). Let U be a (classic universal) set. A fuzzy subset $\hat{u}$ of U is defined by a function $\varphi_{\hat{u}}$, called the **membership function** of $\hat{u}$.

$$\varphi_{\hat{u}} : U \longrightarrow [0, 1].$$

The value of $\varphi_{\hat{u}}(x) \in [0, 1]$ indicates the degree to which the element x of U belongs to the fuzzy set $\hat{u}$.

Definition 2. (*Support*). The classic subset of U

$$\text{supp}\hat{u} = \{x \in U : \varphi_{\hat{u}}(x) > 0\}$$

is called the **support** of $\hat{u}$.

The support of $\hat{u}$ has a fundamental role in the interrelation between classical and fuzzy set theory. Interestingly, unlike fuzzy subsets, a support is a crisp set.

Definition 3. (α -cut). Let $\hat{u}$ be a fuzzy subset of U $\alpha \in [0, 1]$. The α -cut of the subset $\hat{u}$ is the set $[\hat{u}]^\alpha$ of U defined by

$$[\hat{u}]^\alpha = \{x \in U : \varphi_{\hat{u}}(x) \geq \alpha\} \quad \text{para} \quad 0 < \alpha \leq 1$$

and $[\hat{u}]^0 = \overline{\text{supp}\hat{u}}$.

Definition 4. (Zadeh's Extension Principle). Let f be a function such that $f : X \rightarrow Z$ and let $\hat{u}$ be a fuzzy subset of X . **Zadeh's extension** of f is the function $\hat{f}$ which maps to $\hat{u}$ gives us the fuzzy subset $\hat{f}(\hat{u})$ of Z with the membership function given by

$$\varphi_{\hat{f}(\hat{u})}(z) = \begin{cases} \sup_{x=f^{-1}(z)} \varphi_{\hat{u}}(x) & \text{se } f^{-1}(z) \neq \emptyset \\ 0 & \text{se } f^{-1}(z) = \emptyset \end{cases} \quad (2.1)$$

where $f^{-1}(z) = \{x : f(x) = z\}$ is the pre-image of z .

Note that if f is bijective, then the membership function of $\hat{f}(\hat{u})$ is given as follows

$$\varphi_{\hat{f}(\hat{u})}(z) = \sup_{x:f(x)=z} \varphi_{\hat{u}}(x) = \varphi_{\hat{u}}(f^{-1}(z))$$

where f^{-1} is the inverse function of f .

Definition 5. (Fuzzy Number). A fuzzy subset $\hat{u}$ is called a fuzzy number when the universal set on which $\varphi_{\hat{u}}$ is defined is the set of all real numbers $\mathbb{R}$ and satisfies the following conditions:

- (i) all the α -levels of $\hat{u}$ are not empty for $0 \leq \alpha \leq 1$;
- (ii) all the α -levels of $\hat{u}$ are closed intervals of $\mathbb{R}$;
- (iii) $\text{supp } \hat{u} = \{x \in \mathbb{R} : \varphi_{\hat{u}}(x) > 0\}$ is bounded.

Definition 6. A fuzzy number $\hat{u}$ is said to be **triangular** if its membership function is given by

$$\varphi_{\hat{u}}(x) = \begin{cases} 0 & \text{se } x \leq a \\ \frac{x-a}{b-a} & \text{se } a \leq x \leq b \\ \frac{x-c}{c-b} & \text{se } b \leq x \leq c \\ 0 & \text{se } x \geq c \end{cases} \quad (2.2)$$

where a, b, c are given numbers. Let us denote a triangular fuzzy number by (a, b, c) .

The definitions 7 and 8 present two partial order relations for real intervals. These two relations will be used to define a partial order relation for fuzzy numbers.

Definition 7. Let $[a, b]$ e $[c, d]$ closed intervals. We say that $[c, d] \preceq [a, b]$ if and only if, $c \leq a$ e $d \leq b$ (Order of Kulish-Miranker).

Definition 8. Let $[a, b]$ e $[c, d]$ closed intervals. We say that $[c, d] \prec [a, b]$ if and only if,

$$(c < a, d \leq b) \text{ or } (c \leq a, d < b).$$

Definition 9 establishes a relation of partial order between fuzzy numbers, and is this relation of order that we are using to optimize a fuzzy functional.

Definition 9. Let $\hat{u}$ and $\hat{v}$ fuzzy numbers. We say that $\hat{u} \preceq \hat{v}$ ($\hat{u} \prec \hat{v}$), if and only if, $[\hat{u}]^\alpha \preceq [\hat{v}]^\alpha$ ($[\hat{u}]^\alpha \prec [\hat{v}]^\alpha$), $\forall \alpha \in [0, 1]$.

Theorem 1. (Nguyen's Theorem) Let $f : X \rightarrow Z$ be a continuous function, $X, Y \in \mathbb{R}$. and $\hat{u}$ a fuzzy subset of X . So for all $\alpha \in [0, 1]$

$$\left[\widehat{f(\hat{u})} \right]^\alpha = f([\hat{u}]^\alpha).$$

According to [2] and [10], Nguyen's Theorem is also valid for Hausdorff topological spaces, in particular when $\mathbb{X} = \mathbb{R}^2$, as given below.

Theorem 2. Let $\mathbb{X}$ and $\mathbb{Y}$ Hausdorff spaces. If $f : \mathbb{X} \rightarrow \mathbb{Y}$ is continuous, then the extension of Zadeh $\widehat{f} : \mathcal{F}_{\mathcal{K}}(\mathbb{X}) \rightarrow \mathcal{F}_{\mathcal{K}}(\mathbb{Y})$ is well defined and for all $\alpha \in [0, 1]$

$$\left[\widehat{f(\hat{u})} \right]^\alpha = f([\hat{u}]^\alpha).$$

where $\mathcal{F}_{\mathcal{K}}(\mathbb{X})$ denotes the family of all fuzzy subsets of $\mathbb{X}$ with compact and non-empty α -level.

Definition 10. Let x_λ be a real function, continuously parameterized by $\lambda \in [\lambda_L, \lambda_R] \subset \mathbb{R}$ and differentiable for each fixed $\lambda \in [\lambda_L, \lambda_R]$. We define the derivative of the extension of $x_\lambda(t)$ in $[\lambda_L, \lambda_R]$ as

$$\frac{d\widehat{x_\lambda}(t)}{dt} = \frac{\widehat{dx_\lambda}(t)}{dt} \quad (2.3)$$

where, $\widehat{\lambda} \in \mathcal{F}(\lambda_L, \lambda_R]$ and $\frac{\widehat{dx_\lambda}(t)}{dt}$ is Zadeh's extension of $\frac{dx_\lambda(t)}{dt}$ in λ .

Definition 11. Let x_λ be a real function, continuously parameterized by $\lambda \in [\lambda_L, \lambda_R] \subset \mathbb{R}$ and integrable for each fixed $\lambda \in [\lambda_L, \lambda_R]$. We define the integral of the extension of $x_\lambda(t)$ in $[\lambda_L, \lambda_R]$ as

$$\int_a^b \widehat{x_\lambda}(t) dt = \int_a^b \widehat{x_\lambda}(t) dt \quad (2.4)$$

where, $\widehat{\lambda} \in \mathcal{F}([\lambda_L, \lambda_R])$ and $\int_a^b \widehat{x_\lambda}(t) dt$ is Zadeh's extension of $\int_a^b x_\lambda(t) dt$ in λ .

Theorem 3 establishes necessary and sufficient conditions of optimality for problems of variational calculus whose objective functional is convex. These conditions sum up the Euler-Lagrange Equation (2.5) together with a condition of transversality (2.6).

Theorem 3. (See [14]) Let D be a domain in $\mathbb{R}^2$ and for given x_0 , set

$$\mathcal{D} = \{x \in \mathcal{C}^1[t_0, t_f] : x(t_0) = x_0; (x(t), \dot{x}(t)) \in D\}.$$

If $F(t, x, z)$ is (strongly) convex on $[t_0, t_f] \times D$, then

$$J(x) = \int_{t_0}^{t_f} F(t, x, \dot{x}) dt$$

is (strictly) convex on D . Hence each $x \in D$ for which

$$\frac{\partial F}{\partial x}(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}}(t, x^*(t), \dot{x}^*(t)) \right) = 0, \quad (2.5)$$

with the initial condition $x(t_0) = x_0$ and the transversality condition

$$\frac{\partial F}{\partial \dot{x}}(t_f, x(t_f), \dot{x}^*(t_f)) = 0. \quad (2.6)$$

on (t_0, t_f) , minimizes J on D (uniquely).

Remark 4.

- $\dot{x}(t)$ denotes the derivative of x with respect to t .
- Equation (2.5) is called Euler-Lagrange Equation.

The next theorem establishes sufficient conditions ensuring the monotonicity of the solution with respect to the initial condition. A detailed proof can be found in [4].

Theorem 5. Consider the following variational problem:

$$\min_{x \in C^1([0, t_f])} J(x) = \int_0^{t_f} (ax(t)^2 + bx(t)\dot{x}(t) + c\dot{x}(t)^2) dt, \quad (2.7)$$

subject to the boundary conditions

$$x(0) = x_0, \quad x(t_f) \text{ is free.}$$

If one of the following four conditions holds:

- If a and c are nonzero, they have the same sign and $b = 0$; or
- If a and c are nonzero, they have the same sign, $b \neq 0$ and $|b| \leq 2\sqrt{ac}$; or
- If $a = 0$, $c \neq 0$, and $0 < \frac{bt}{bt_f - 2c} < 1$, for all $t \in (0, t_f)$; or
- If a and c are nonzero, have opposite signs, $b \neq 0$, and

$$0 < t \sqrt{\frac{c}{a}} < \frac{\pi}{2}, \quad \text{for all } t \in (0, t_f),$$

then the solution $x(t; x_0)$ of problem (2.14), whenever it exists, is monotone increasing with respect to the initial condition.

3 FUZZY FUNCTIONAL

The formulation of variational problems under uncertainty requires an appropriate extension of the classical notions of functional and extremum. When the initial condition or system parameters are modeled by fuzzy numbers, the value of the functional also becomes an imprecise quantity, whose evaluation must respect the structure of fuzzy logic. Thus, to assign meaning to the functional associated with the variational problem, it is necessary to understand how a mapping defined on function spaces can be extended to the fuzzy setting.

In this section, we present the procedure for computing the fuzzy value of a functional associated with a variational problem with fuzzy initial data. The main idea is to decompose the classical functional into a composition of real-valued functions and mappings between function spaces, and then apply Zadeh's extension principle to each stage of the composition. This allows us to obtain the fuzzy functional value directly through its α -cuts, while preserving essential properties such as continuity, compactness, and connectedness.

With this approach, we show that the resulting mapping is well defined and yields a fuzzy number, enabling a rigorous treatment of variational problems in nondeterministic environments.

Given the problem

$$\min_{x \in C^1([t_0, t_f])} J(x) = \int_{t_0}^{t_f} F(t, x, \dot{x}) dt \quad (3.1)$$

$$x(0) = \hat{x}_0 \quad x(t_f) \text{ is free,}$$

where F is a real function, $\hat{x}_0$ is a fuzzy number and $\mathcal{C}^1[t_0, t_f]$ is the space of continuously differentiable functions from the closed interval $[t_0, t_f]$ to $\mathbb{R}$.

In this section we will see how to calculate the value of the fuzzy functional (3.1). In summary, we can get this value by doing a Zadeh's extension of compound functions as follows

$$x_0 \xrightarrow{\phi_1} x_{x_0}(\cdot) \xrightarrow{\phi_2} F(t, x, \dot{x}) \xrightarrow{\phi_3} \int_{t_0}^{t_f} F(t, x, \dot{x}) dt = J(x(\cdot, x_0)). \quad (3.2)$$

Where,

$$\phi_1 : [x_0^L, x_0^R] \subset \mathbb{R} \rightarrow \mathcal{C}^1[t_0, t_f], \quad (3.3)$$

$$\phi_2 : \mathcal{C}^1[t_0, t_f] \rightarrow \mathcal{C}[t_0, t_f] \quad (3.4)$$

and

$$\phi_3 : \mathcal{C}[t_0, t_f] \rightarrow \mathbb{R} \quad (3.5)$$

Thus, by applying the Zadeh's extension in each of the functions, we obtain:

$$\hat{x}_0 \xrightarrow{\hat{\phi}_1} \hat{x}_{\hat{x}_0}(\cdot) \xrightarrow{\hat{\phi}_2} \hat{F}(t, \hat{x}, \hat{\dot{x}}) \xrightarrow{\hat{\phi}_3} \widehat{\int_{t_0}^{t_f} \hat{F}(t, \hat{x}, \hat{\dot{x}}) dt} = \hat{J}(\hat{x}(\cdot, \hat{x}_0)). \quad (3.6)$$

Proposition 1. Let $\phi_1 : \mathbb{R} \rightarrow \mathcal{C}^1[t_0, t_f]$ be a continuous function and $\hat{x}_0$ a fuzzy number. Then the Zadeh's extension $\hat{\phi} : \mathcal{F}_{\mathbb{R}} \rightarrow \mathcal{F}_{\mathcal{H}}(\mathcal{C}^1[t_0, t_f])$ is well defined and for all $\alpha \in [0, 1]$

$$[\hat{x}(t, \hat{x}_0)]^\alpha = \hat{x}(t, [\hat{x}_0]^\alpha).$$

Proof. Initially note that $\mathbb{R}$ and $\mathcal{C}^1([t_0, t_f])$ are Hausdorff spaces and we have ϕ_1 is a continuous application. The α -levels of the elements of $\mathcal{F}_{\mathbb{R}}$ are compact sets, since they are closed intervals of $\mathbb{R}$. For us to use Theorem 1 we must show that $\hat{\phi}_1$ is well defined, the α -levels of $\mathcal{F}_{\mathcal{H}}(\mathcal{C}^1[t_0, t_f])$ are compact and non-empty.

For each $0 \leq \alpha \leq 1$, we show that $[\hat{\phi}(\hat{x}_0)]^\alpha$ is compact and not empty. Note that to show that $[\hat{x}(t, \hat{x}_0)]^\alpha \neq \emptyset$ requires just to show that $[\hat{x}(t, \hat{x}_0)]^1 \neq \emptyset$. In fact, since $\hat{x}_0$ is a fuzzy number, then $[\hat{x}_0]^1 \neq \emptyset$ and since each function in $[\hat{x}(t, \hat{x}_0)]^1$ is obtained from the initial condition at $[\hat{x}(t, \hat{x}_0)]^1 \neq \emptyset$.

In order to show the compactness, take X_n a sequence in $[\hat{x}(t, \hat{x}_0)]^\alpha$, as each element of $[\hat{x}(t, \hat{x}_0)]^\alpha$ is a function $x(t, x_0) \in C_1$, which depends on the initial condition, so we can write $X_n = x(t, x_{0n})$, where x_{0n} is a sequence in $\text{supp}x_0 = [x_0^L, x_0^R]$, as is an interval closed in $\mathbb{R}$, then x_{0n} has a subsequence x_{0nk} , which converges, say to $\bar{x}_0$.

Define the subsequence $X_{nk} = x(t, x_{0nk})$, then using the continuous dependence on the initial condition we have

$$\lim X_{nk} = \lim X(t, x_{0nk}) = X(t, \lim x_{0nk}) = X(t, \bar{x}_0).$$

This shows that X_n has a subsequence X_{nk} that converges, so $[\hat{x}(t, \hat{x}_0)]^\alpha$ is compact $\forall 0 \leq \alpha \leq 1$. $\square$

Proposition 2. Let ϕ_2 e ϕ_3 be defined in 3.4 and 3.5, continuous functions and $\hat{x}_0$ a fuzzy number. Being $\hat{\phi}_2$ e $\hat{\phi}_3$ the Zadeh's extension of the ϕ_2 e ϕ_3 , respectively, then

$$[\hat{J}(\hat{x}(t, \hat{x}_0))]^\alpha = J([\hat{x}(t, \hat{x}_0)]^\alpha).$$

Proof. The proof of this proposition follows the same lines as the 1. It is enough to show that $[\hat{J}(\hat{x}(t, \hat{x}_0))]^\alpha$ are compact and non-empty sets $0 \leq \alpha \leq 1$ and use Theorem 2. $\square$

Given the propositions 1 and 2 let us show that $\hat{J}(\hat{x}(t, \hat{x}_0))$ is a fuzzy number, since The order relation to be used is for fuzzy numbers.

Theorem 1. Let $\hat{x}_0 \in \mathcal{F}_{\mathbb{R}}$ e ϕ_1, ϕ_2 e ϕ_3 be continuous functions, then $\hat{J}(\hat{x})$ defined by the composition of Zadeh extensions below

$$\hat{x}_0 \xrightarrow{\hat{\phi}_1} \hat{x}_{\hat{x}_0}(t) \xrightarrow{\hat{\phi}_2} \hat{F}(t, \hat{x}, \hat{x}) \xrightarrow{\hat{\phi}_3} \int_{t_0}^{t_f} \hat{F}(t, \hat{x}, \hat{x}) dt = \hat{J}(\hat{x}(t, \hat{x}_0)).$$

is the fuzzy number.

Proof.

1. Since $\hat{x}_0$ a fuzzy number, by propositions 1 and 2, $\left[\hat{J}(\hat{x}(t), \hat{x}_0)\right]^\alpha = J([\hat{x}(t), \hat{x}_0]^\alpha) = J(x(t, [\hat{x}_0]^\alpha))$, then $[\hat{x}_0]^\alpha$ are closed intervals of the $\mathbb{R}$, then closed and connected sets, since ϕ_1, ϕ_2 , and ϕ_3 are continuous functions and the image of closed and connected by continuous functions are connected sets ([12]). then $\left[\hat{J}(\hat{x}(t), \hat{x}_0)\right]^\alpha$ are connected and closed sets $\forall 0 \leq \alpha \leq 1$. Since the only closed and connected sets of $\mathbb{R}$ are the closed intervals, then $\left[\hat{J}(\hat{x}(t), \hat{x}_0)\right]^\alpha$ are closed intervals $\forall 0 \leq \alpha \leq 1$.

2. Still, by propositions 1 e 2

$$\left[\hat{J}(\hat{x}(t), \hat{x}_0)\right]^1 = J([\hat{x}(t), \hat{x}_0]^1) = J(x(t, [x_0]^1))$$

$[x_0]^1 \neq \emptyset$ because $\hat{x}_0$ is a fuzzy number, so $J(x(t, [x_0]^1)) \neq \emptyset$ e consequently α -níveis are non-empty, $\forall 0 \leq \alpha \leq 1$.

3. $\text{supp}(\hat{J}(\hat{x}(t), \hat{x}_0)) \subset \overline{\text{supp}}(\hat{J}(\hat{x}(t), \hat{x}_0)) = \left[\hat{J}(\hat{x}(t), \hat{x}_0)\right]^0$
 $= J([\hat{x}(t), \hat{x}_0]^0) = J(x(t, [x_0]^0))$

since $\hat{x}_0$ a fuzzy number, então $[x_0]^\alpha$ are closed intervals, thus bounded sets, and sin ϕ_1, ϕ_2 e ϕ_3 3 are continuous functions and the image of bounded sets by continuous functions are bounded se ([12]), then $J(x(t, [x_0]^0))$ are bounded. So, $\text{supp}(\hat{J}(\hat{x}(t), \hat{x}_0)) \subset J(x(t, [x_0]^0))$ is bounded.

4. Again, as $\hat{x}_0$ is a fuzzy number then, $\forall 0 \leq \alpha \leq \beta \leq 1 \Rightarrow [\hat{x}_0]^\beta \subset [\hat{x}_0]^\alpha$. So, using the propositions 1 and 2, we have

$$\begin{aligned} 0 \leq \alpha \leq \beta \leq 1 &\Leftrightarrow [\hat{x}_0]^\beta \subset [\hat{x}_0]^\alpha \Leftrightarrow [\hat{x}(t), \hat{x}_0]^\beta \subset [\hat{x}(t), \hat{x}_0]^\alpha \\ &\Leftrightarrow [\hat{J}(\hat{x}(t), \hat{x}_0)]^\beta \subset [\hat{J}(\hat{x}(t), \hat{x}_0)]^\alpha. \end{aligned}$$

So, $\hat{J}(\hat{x})$ is a fuzzy number.

□

4 RESULTS

Definition 12. We define feasible set (or admissible) for the problem (3.1) as the set of all fuzzy subsets of functions $\hat{x}(t, \hat{x}_0)$ such that there is a continuous application that associates elements of $[\hat{x}_0]^0$ to the elements of $[\hat{x}(t, \hat{x}_0)]^0$.

Theorem 1. Consider the problem

$$\min_{x \in C^1([0, t_f])} J(x) = \int_0^{t_f} [F(t, x, \dot{x})] dt \quad (4.1)$$

$$x(0) = \hat{x}_0 \text{ e } x(t_f) \text{ is free}$$

with $\hat{x}_0 \in \mathcal{F}_{\mathbb{R}}$. If

1. $F(t, x, \dot{x})$ is continuous with respect to all variables;
2. The functional $J(x)$ is convex;
3. The solution of the classical problem is continuous with respect to and monotonic non-decreasing with respect to the initial condition.

Then $\hat{x}^*(t, \hat{x}_0)$ is minimizer of Problem 4.1 for all $\hat{x}(t, \hat{x}_0)$ belonging to the feasible set of 4.1 according to a minimizer definition given in 2.9.

Proof. Note that a solution of the classical problem $x(t, x_0)$ is monotonically increasing with respect the initial condition, then we can make an extension of Zadeh with respect to x_0 . We must show that $\hat{J}(\hat{x}^*(t, \hat{x}_0)) \preceq \hat{J}(\hat{x}(t, \hat{x}_0))$, for all $\hat{x}(t, \hat{x}_0)$ belonging to the feasible set of (4.1) and satisfying the boundary conditions.

First, we must verify that $\hat{J}(\hat{x}^*(t, \hat{x}_0))$ and $\hat{J}(\hat{x}(t, \hat{x}_0))$ are fuzzy numbers, and so we can use the relation of order defined in 2.9. By hypothesis, we have $F(t, x, \cdot)$ is continuous with respect to all variables and that the solution of the classical problem is continuous with respect to the initial condition, by Theorem 1, $\hat{J}(\hat{x}^*(t, \hat{x}_0))$ is a fuzzy number. In addition, any admissible fuzzy solution is supported by a set of admissible solutions that depend continuously on the initial solution, so we can guarantee that $\hat{J}(\hat{x}(t, \hat{x}_0))$ is also a fuzzy number, so we can use the order relation defined in 2.9.

Therefore, we have to show that $\forall \alpha \in [0, 1]$, we have

$$\hat{J}(\hat{x}^*(t, \hat{x}_0))_L^\alpha \preceq \hat{J}(\hat{x}(t, \hat{x}_0))_L^\alpha \text{ and } \hat{J}(\hat{x}^*(t, \hat{x}_0))_R^\alpha \preceq \hat{J}(\hat{x}(t, \hat{x}_0))_R^\alpha \quad (4.2)$$

Let $\hat{x}(t, \hat{x}_0)$ be a function belonging to the admissible set of the problem. Initially we will demonstrate that the first inequality of (4.2) is valid.

Given $\hat{J}(\hat{x}^*(t, \hat{x}_0))$ a fuzzy number, its α -levels are closed intervals, so we have:

$$\hat{J}(\hat{x}^*(t, \hat{x}_0))_L^\alpha = \min \left[\hat{J}(\hat{x}^*(t, \hat{x}_0)) \right]^\alpha \quad (4.3)$$

Using the proposition 2 we have

$$\min \left[\hat{J}(\hat{x}^*(t, \hat{x}_0)) \right]^\alpha = \min J([\hat{x}^*(t, \hat{x}_0)]^\alpha) \quad (4.4)$$

Using the same rationale for $\hat{x}(t, \hat{x}_0)$ we have to

$$\min \left[\hat{J}(\hat{x}(t, \hat{x}_0)) \right]^\alpha = \min J([\hat{x}(t, \hat{x}_0)]^\alpha) \quad (4.5)$$

Since the solution and any function belonging to the feasible set of (4.1) are continuously dependent on the initial condition then by the proposition 1 $[\hat{x}^*(t, \hat{x}_0)]^\alpha = x^*(t, [x_0]^\alpha)$ and $[\hat{x}(t, \hat{x}_0)]^\alpha = x(t, [x_0]^\alpha)$, thereby

$$\hat{J}(\hat{x}^*(t, \hat{x}_0))_L^\alpha = \min \left[\hat{J}(\hat{x}^*(t, \hat{x}_0)) \right]^\alpha = \min J([\hat{x}^*(t, \hat{x}_0)]^\alpha) = \min J([x^*(t, [x_0]^\alpha)]) \quad (4.6)$$

and

$$\widehat{J}(\widehat{x}(t, \widehat{x}_0))_L^\alpha = \min \left[\widehat{J}(\widehat{x}(t, \widehat{x}_0)) \right]^\alpha = \min J([\widehat{x}(t, \widehat{x}_0)]^\alpha) = \min J([x(t, [x_0]^\alpha)]) \quad (4.7)$$

Let $x^*(t, x_0^*) \in x(t, \bar{x}_0)$ be the arguments that minimize respectively $J([x^*(t, [x_0]^\alpha)])$ and $J([x(t, [x_0]^\alpha)])$, note that $\bar{x}_0 \in [x_0]^\alpha$, soon $x(t, \bar{x}_0) \in [\widehat{x}^*(t, \widehat{x}_0)]^\alpha$ and we can write

$$J(x^*(t, x_0^*)) \leq J(x^*(t, \bar{x}_0)) \quad (4.8)$$

In addition, since the functional is convex, $x^*(t, \bar{x}_0)$ is the global optimal solution for the classical problem with initial condition $\bar{x}_0$, hence

$$J(x^*(t, \bar{x}_0)) \leq J(x(t, \bar{x}_0)) \quad (4.9)$$

So,

$$\widehat{J}(\widehat{x}^*(t, \widehat{x}_0))_L^\alpha = J(x^*(t, x_0^*)) \leq J(x^*(t, \bar{x}_0)) \leq J(x(t, \bar{x}_0)) = \widehat{J}(\widehat{x}(t, \widehat{x}_0))_L^\alpha \quad (4.10)$$

The second inequality is proved analogously. $\square$

Example 4.1. In this example, we solve the following minimization problem:

$$\begin{aligned} \min_{x \in C^1([0,2])} J(x) &= \int_0^2 [x(t)^2 + \dot{x}(t)^2] dt \\ x(0) &= \widehat{x}_0 \text{ and } x(2) \text{ is free,} \end{aligned} \quad (4.11)$$

where $\widehat{x}_0$ is the triangular fuzzy number $(1; 1.5; 2)$.

First, note that $J(x)$ is a quadratic functional and, comparing it with problem (2.7), we have $a = c = 1$ and $b = 0$. Therefore, by item (a) of Theorem 5, the solution of problem 4.11, whenever it exists, is monotone increasing with respect to the initial condition.

By Theorem 3, the necessary condition of optimality for the classical problem is the solution of the Euler–Lagrange equation:

$$F_x(t, x(t), \dot{x}(t)) - \frac{d}{dt} (F_{\dot{x}}(t, x(t), \dot{x}(t))) = 0, \quad (4.12)$$

$$2x(t) - \frac{d}{dt} (2\dot{x}(t)) = 0, \quad (4.13)$$

$$x(t) - \ddot{x}(t) = 0. \quad (4.14)$$

Thus, a necessary condition for $x^*(t)$ to be a solution to problem 4.11 is that it satisfies the second–order differential equation

$$x(t) - \ddot{x}(t) = 0,$$

whose general solution is

$$x^*(t) = c_1 e^t + c_2 e^{-t}.$$

To determine the constants c_1 and c_2 , we use the initial condition $x(0) = x_0$:

$$c_1 + c_2 = x_0,$$

together with the natural boundary condition

$$\begin{aligned} F_{\dot{x}}(t_f, x(t_f), \dot{x}(t_f)) &= 0, \\ c_1 e^2 - c_2 e^{-2} &= 0. \end{aligned}$$

Therefore,

$$c_1 = \frac{x_0}{1 + e^4}, \quad c_2 = \frac{x_0 e^4}{1 + e^4},$$

and consequently, the classical solution depending on the initial condition is

$$x^*(t) = \frac{x_0}{1 + e^4} (e^t + e^{4-t}). \quad (4.15)$$

Note that:

- $F(t, x, \dot{x}) = x(t)^2 + \dot{x}(t)^2$ **is continuous with respect to all variables.**
- **The classical problem is convex.**

Indeed, in this case $F(t, x, \dot{x}) = x^2 + \dot{x}^2$. Therefore,

$$F_{xx} = 2, \quad F_{x\dot{x}} = 0 = F_{\dot{x}x}, \quad F_{\dot{x}\dot{x}} = 2,$$

and the Hessian

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is positive definite, which implies that $F(t, x, \dot{x})$ is strictly convex (and hence convex). Consequently, the classical problem associated with 4.11 is strictly convex. By Theorem ??, problem 4.11 has a unique global solution, namely the one given in (4.15).

- $x^*(t) = \frac{x_0}{1 + e^4} (e^t + e^{4-t})$ **is continuous on $[x_0^L, x_0^R] = [1, 2]$ and monotone increasing with respect to the initial condition.**

Indeed, $x^*(t)$ is linear in x_0 , and therefore continuous for all $x_0 \in [1, 2]$. Moreover, we have already shown that the solution is monotone increasing with respect to the initial condition.

Therefore, all hypotheses of Theorem 1 are satisfied, and we can conclude that the Zadeh extension of $x^*(t)$, with respect to x_0 , is a fuzzy set of functions that solves the corresponding fuzzy variational problem. Figure 1 illustrates the solution.

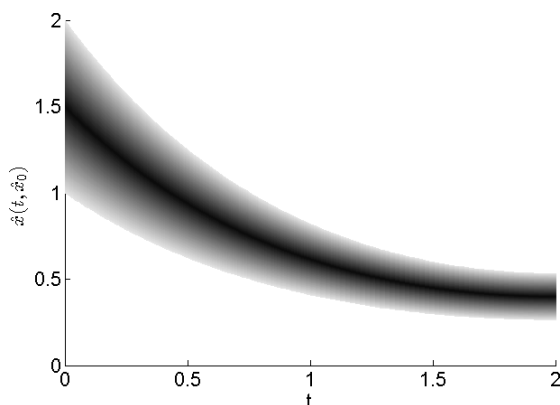


Figure 1: Optimal solution of 4.11 with fuzzy initial condition $\hat{x}_0 = (1; 1.5; 2)$.

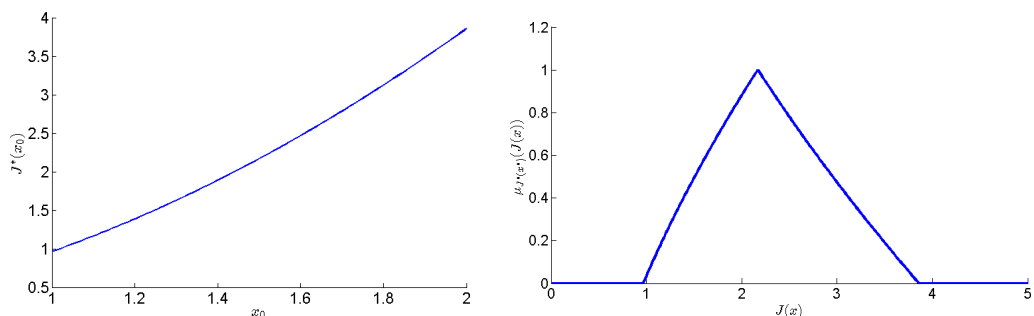
To compute the optimal value of the fuzzy functional, we first compute the optimal value of the classical functional and then apply Zadeh's extension principle with respect to the initial condition x_0 .

$$J(x^*(t, x_0)) = \int_0^2 \left[\frac{x_0}{1+e^4} (e^t + e^{4-t}) \right]^2 + \left[\frac{x_0}{1+e^4} (e^t - e^{4-t}) \right]^2 dt \quad (4.16)$$

$$= \int_0^2 \frac{2x_0^2}{(1+e^4)^2} (e^{2t} + e^{8-2t}) dt \quad (4.17)$$

$$= \frac{x_0^2}{(1+e^4)^2} (e^8 - 1). \quad (4.18)$$

Note that the optimal value of the fuzzy functional depends on the initial condition. We can see the corresponding optimal value of the fuzzy functional in Figure 2.



Evolution of the optimal value with respect to the initial condition

Optimal value of the fuzzy functional

Figure 2

We now perform comparisons to verify that the fuzzy value of the functional for the optimal solution $\hat{x}^*(t, \hat{x}_0)$ is smaller than the value of the functional evaluated at any admissible fuzzy function $\hat{x}(t, \hat{x}_0)$.

- Comparison 1:

Consider the following admissible function, dependent on the initial condition:

$$x(t, x_0) = x_0 \left(\frac{e^2}{1 + e^4} - \frac{1}{2} \right) t + x_0 \quad (4.19)$$

Thus, the value of the objective functional, dependent on the initial condition, is $J(u) = 6.12784x_0^2$. Applying Zadeh's extension principle to x , we obtain the fuzzy solution shown in Figure 3.

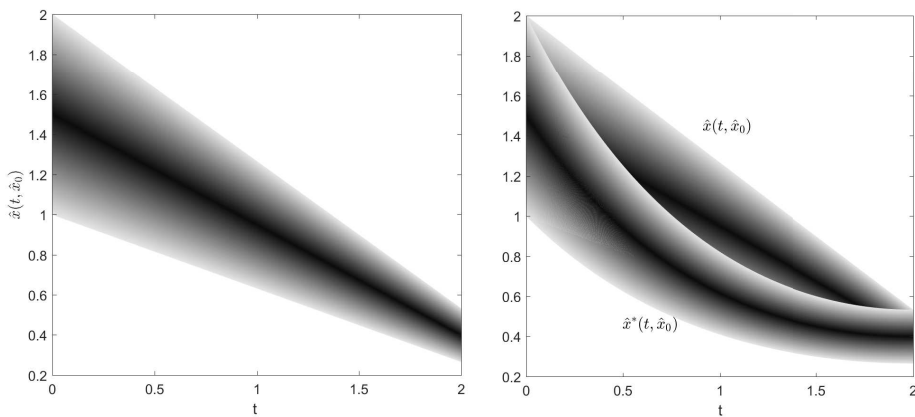


Figure 3: Plot of $\hat{x}(t, \hat{x}_0)$ and comparison between $\hat{x}^*(t, \hat{x}_0)$ and $\hat{x}(t, \hat{x}_0)$.

Figure 4 shows the comparison between $\hat{J}(\hat{x}^*)$ and $\hat{J}(\hat{x})$, confirming that, indeed, $\hat{J}(\hat{x}^*) \prec \hat{J}(\hat{x})$.

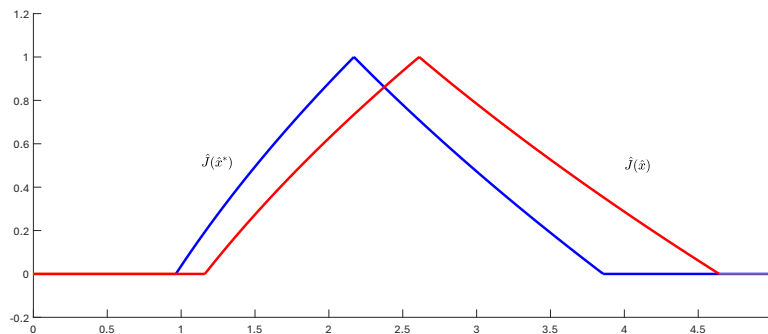


Figure 4: Comparison between $\hat{J}(\hat{x}^*)$ and $\hat{J}(\hat{x})$.

• Comparison 2:

We now compare with a function that satisfies the boundary conditions and oscillates around the initial function. To this end, consider the following feasible solution, dependent on the initial condition:

$$x(t, x_0) = \left[\frac{x_0}{1 + e^4} (e^t + e^{4-t}) \right] \left[\frac{\sin(10\pi t)}{10} + 1 \right] \quad (4.20)$$

The value of the objective functional is $J(x) = 829.561x_0^2$. Applying Zadeh's extension principle to x , we obtain the fuzzy solution shown in Figure 5.

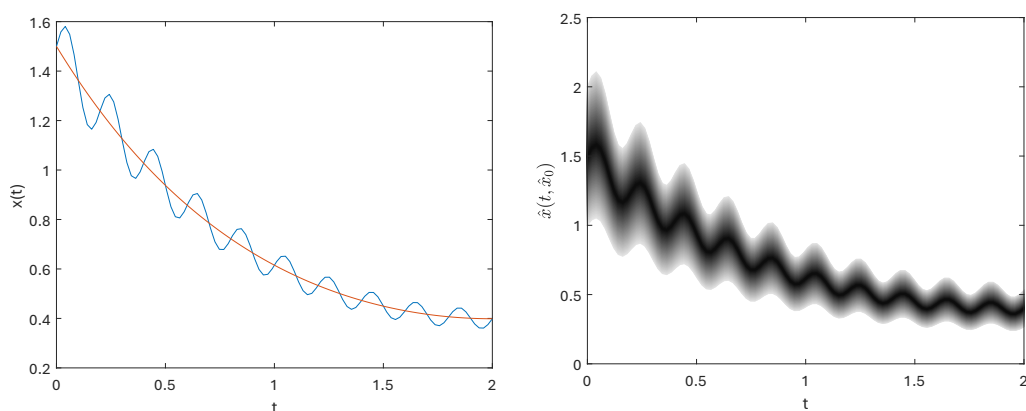


Figure 5: Comparison between the optimal solution with $x_0 = 1.5$ and a perturbation $\hat{x}(t, \hat{x}_0)$, with $\hat{x}_0 = (1; 1.5; 2)$.

Figure 6 shows the membership function of the fuzzy numbers $\hat{J}(\hat{x})$ and $\hat{J}(\hat{x}^*)$, respectively, confirming that $\hat{J}(\hat{x}^*) \prec \hat{J}(\hat{x})$.

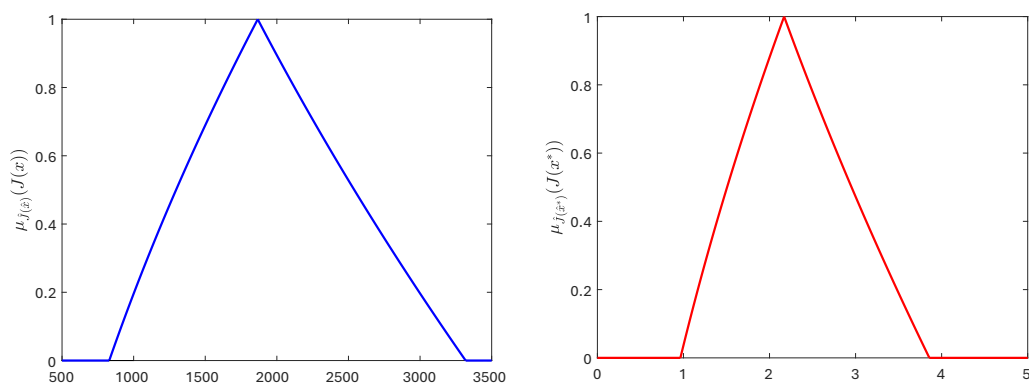


Figure 6: Comparison between $\hat{J}(\hat{x})$ and $\hat{J}(\hat{x}^*)$.

5 CONCLUSIONS

We study Problems of the variational calculus with fuzzy initial condition and free final condition. Following the idea developed in [5, 6], we used Zadeh's extension of the classical solutions with respect to initial condition and, thereby, the functional was also obtained via extension. Under some hypotheses we show that the fuzzy value of the functional is a fuzzy number and with this, we use a partial order relation to show that under certain conditions the Zadeh's extension of the classical solution minimizes the fuzzy functional. The hypothesis of functional convexity was required so that we could guarantee global optimum solution. A example is done to illustrate the result.

Data availability

Do not apply.

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REFERENCES

- [1] B. Bede. "Mathematics of Fuzzy Sets and Fuzzy Logic". Springer (2013).
- [2] M.S. Cecconello. "Sistemas Dinâmicos em Espaços Métricos Fuzzy – Aplicações em Biomatemática". Ph.D. thesis, Universidade Estadual de Campinas – IMECC, Campinas, Brasil (2010).
- [3] L.C. de Barros, R.C. Bassanezi & W.A. Lodwick. "A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics". Springer-Verlag Berlin (2016).
- [4] J.D.S. de Oliveira. "Sobre cálculo variacional e controle ótimo fuzzy com aplicações: On fuzzy variational calculus and fuzzy optimal control with applications". Ph.D. thesis, [sn] (2018).
- [5] M.M. Diniz. "Otimização de funções, funcionais e controle fuzzy". Ph.D. thesis, Instituto de Matemática, Estatística e Computação Científica (IMECC), UNICAMP (2016).
- [6] M.M. Diniz, L.T. Gomes & R.C. Bassanezi. Solution to Convex Variational Problems with Fuzzy Initial Condition Using Zadeh's Extension. In "North American Fuzzy Information Processing Society Annual Conference". Springer (2018), p. 431–438.
- [7] O.S. Fard, A.H. Borzabadi & M. Heidari. On Fuzzy Euler–Lagrange Equations. *Ann. Fuzzy Math. Inform.*, **7**(2) (2014), 447–461.
- [8] O.S. Fard & M.S. Zadeh. Note on Necessary Optimality Conditions for Fuzzy Variational Problems. *J. Adv. Res. Dyn. Control Syst.*, **4**(2) (2012), 1–9.
- [9] B. Farhadinia. Necessary Optimality Conditions for Fuzzy Variational Problems. In "North American Fuzzy Information Processing Society Annual Conference". Springer (2018), p. 431–438.

- [10] L.T. Gomes, L.C. de Barros & B. Bede. “Fuzzy Differential Equations in Various Approaches”. Springer International Publishing, New York (2015).
- [11] G. Klir & B. Yuan. “Fuzzy Sets and Fuzzy Logic”. Prentice Hall, New Jersey, 4 ed. (1995).
- [12] E. Kreyszig. “Introductory Functional Analysis with Applications”. Wiley, New York (1978).
- [13] J. Soolaky, O.S. Fard & A.H. Borzabadi. Generalized Euler–Lagrange Equations for Fuzzy Fractional Variational Calculus. *Mathematical Communications*, **21**(2) (2016), 199–218.
- [14] J.L. Troutman. “Variational Calculus and Optimal Control: Optimization with Elementary Convexity”. Springer Science and Business Media (2012).
- [15] L. Zadeh. Fuzzy Sets. *Information and Control*, **8**(3) (1965), 338–353.

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