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Optimal Control of Neutral Functional-Differential Inclusions Linear in Velocities

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Abstract. This paper studies optimal control problems for dynamical systems governed by neutral functional-differential inclusions that linearly depend on delayed velocity variables. Developing the method of discrete approximations, we derive new necessary optimality conditions for such problems in both Euler-Lagrange and Hamiltonian forms. The results obtained are expressed in terms of advanced generalized differential constructions in variational analysis.

Key words. Optimal control, neutral functional-differential inclusions, discrete approximations, generalized differentiation, necessary optimality conditions.

AMS subject classification. 49K24, 49K25, 49J52, 49M25, 90C31.

1. Introduction

This paper considers the optimal control problem (P) formulated as follows: minimize the cost functional:

$$J[x] := \varphi(x(a), x(b)) + \int_a^b f(x(t), x(t-\Delta), t)\, dt \tag{1.1}$$

over arcs $x : [a-\Delta, b] \to I\!R^n$, which are absolutely continuous on $[a-\Delta, a)$ and $[a, b]$ ($t = a$ could be a point of discontinuity) and satisfy the following neutral functional-differential inclusion:

$$\begin{cases} \dot{x}(t) - A\dot{x}(t-\Delta) \in F(x(t), x(t-\Delta), t) & \text{a.e. } t \in [a, b], \\ x(t) = c(t), \quad t \in [a-\Delta, a), \end{cases} \tag{1.2}$$

with the endpoint constraints

$$(x(a), x(b)) \in \Omega \subset I\!R^{2n}. \tag{1.3}$$

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We always assume that $F : I\!R^n \times I\!R^n \times [a,b] \rightrightarrows I\!R^n$ is a set-valued mapping, that Ω is a closed set, that $\Delta > 0$ is a constant delay, and that A is a $n \times n$ constant matrix.

The main objective of this paper is to derive necessary optimality conditions for problem (P) under natural assumptions on the initial data. For optimal control systems governed by delayed differential inclusions ($A = 0$) necessary optimality conditions have been studied in several papers, particularly by Clarke and Watkins [2], Clarke and Wolenski [3], Minchenko [7], Mordukhovich and Trubnik [12], and Mordukhovich and Wang [13]. Quite recently [14, 15], Mordukhovich and Wang developed first results on optimal control of the so-called *neutral functional-differential inclusions* (i.e., with $A \neq 0$ in (1.2)) given in the *Hale form*

$$\frac{d}{dt}\Big[x(t) - Ax(t-\Delta)\Big] \in F(x(t), x(t-\Delta), t) \quad \text{a.e. } t \in [a,b]$$

that happens to be a simplification of (1.2) and does not allow us to derive appropriate *intrinsic* results for the conventional form (1.2) of neutral systems. The main goal of this paper is to obtain necessary optimality conditions for problem (P) given in the conventional form of neutral functional-differential inclusions, which have not been considered in the literature from the viewpoint of optimality conditions.

The techniques used in this paper are based on the *method of discrete approximations*, which was developed by Mordukhovich [8, 10] for ordinary differential inclusions, then extended by Mordukhovich and Trubnik [12] and Mordukhovich and Wang [13] for delay-differential inclusions, and then by Mordukhovich and Wang [14, 15] for neutral functional-differential inclusions given in the Hale form. This method allows us to construct a *well-posed* parametric family of optimal control problems for approximating systems governed by discrete-time neutral functional-difference inclusions, which can be reduced in turn to problems of nonsmooth mathematical programming with many geometric constraints. To handle such problems, we use generalized differential tools of variational analysis. Finally, passing to the limit from discrete approximations, we obtain necessary optimality conditions for the original continuous-time problem (P). Observe that considering functional-differential inclusions in the general (but linear in velocities) form (1.2) requires a more careful variational analysis from both viewpoints of discrete approximations and deriving necessary optimality conditions. Note also that the linear dependence on velocities in (1.2) is essential in our techniques involving limiting procedures.

The paper is organized as follows. In Section 2 we present some results ensuring the *strong convergence* of optimal solutions in the process of discrete approximation, which play a substantial role in our approach. In Section 3 we review basic constructions and calculus rules of generalized differentiation that are needed to perform a variational analysis of discrete-time and continuous-time systems in the subsequent sections. Section 4 is devoted to necessary optimality conditions for nonconvex discrete-time neutral functional-difference inclusions. The main results on the new Euler-Lagrange and Hamiltonian optimality conditions for the original problem (P) are derived in Section 5 by passing to the limit in the above optimality conditions for discrete-time systems.

The notation in this paper is basically standard. The transposed matrix of A

is denoted by A^*; the symbol $I\!B$ always stands for the closed unit ball of $I\!R^n$, and haus(Ω_1, Ω_2) denotes the Hausdorff distance between two compact sets Ω_1 and Ω_2 in $I\!R^n$. Given a set-valued mapping $F\colon X \rightrightarrows Y$ between finite-dimensional spaces, the *Painlevé-Kuratowski upper/outer limit* of $F(x)$ as $x \to \bar{x}$ is defined by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \big\{ y \in Y \,|\, \exists\, x_k \to \bar{x},\ \exists\, y_k \to y \ \text{ with } \ y_k \in F(x_k), \forall\, k \in I\!N \big\}.$$

Some special symbols are introduced and explained in Section 3. We refer the reader to Mordukhovich [10] and Rockafellar and Wets [17] for additional material and more discussions on generalized differentiation.

2. Convergence of Discrete Approximations

In this section we consider the construction of *well-posed* discrete approximations to problem (P) and present some results on the *strong approximation* and *strong convergence* of optimal solutions.

Let $\bar{x}(t)$ be an arbitrary trajectory. Assume that the set-valued mapping $F(x, y, t)$ is locally bounded, locally Lipschitzian in (x, y) around $\bar{x}(t)$, and Hausdorff continuous a.e. on $[a, b]$. More precisely, we impose the following assumptions throughout the paper:

(H1) There are an open set $U \subset I\!R^n$ and two positive numbers ℓ_F, m_F such that $\bar{x}(t) \in U$ for any $t \in [a - \Delta, b]$, the sets $F(x, y, t)$ are closed, and

$$F(x, y, t) \subset m_F I\!B,$$
$$F(x_1, y_1, t) \subset F(x_2, y_2, t) + \ell_F(|x_1 - x_2| + |y_1 - y_2|)I\!B$$

for all (x, y), (x_1, y_1), $(x_2, y_2) \in U \times U$ and $t \in [a, b]$.

(H2) $F(x, y, t)$ is a.e. Hausdorff continuous on $t \in [a, b]$ uniformly in $U \times U$.

Regarding functions c, φ, and f, we assume that:

(H3) The function $c(\cdot)$ is absolutely continuous on $[a - \Delta, a]$.

(H4) $\varphi(x, y)$ is continuous on $(x, y) \in U \times U$, $f(x, y, t)$ is continuous for a.e. $t \in [a, b]$ uniformly in $(x, y) \in U \times U$, and continuous on $(x, y) \in U \times U$ uniformly in $t \in [a, b]$.

To construct discrete approximations, we replace derivatives in (1.2) by Euler finite differences

$$\dot{x}(t) \approx [x(t + h) - x(t)]/h, \quad \dot{x}(t - \Delta) \approx [x(t + h - \Delta) - x(t - \Delta)]/h.$$

For any natural number N, let $h_N := \Delta/N$ and $t_j := a + jh_N$ for $j = -N, \ldots, k$, where k is a natural number defined by $a + kh_N \le b < a + (k + 1)h_N$. Denote $t_{k+1} := b$. Then the discrete approximations to (1.2) are given as follows:

$$\begin{cases} x_N(t_{j+1}) - Ax_N(t_{j+1} - \Delta) \in x_N(t_j) - Ax_N(t_j - \Delta) \\ \qquad\qquad + h_N F(x_N(t_j), x_N(t_j - \Delta), t_j) \quad \text{for } j = 0, \ldots, k, \\ x_N(t_j) = c(t_j) \quad \text{for } j = -N, \ldots, -1. \end{cases} \tag{2.1}$$

A collection of vectors $\{x_N(t_j) \mid j = -N, \ldots, k+1\}$ satisfying (2.1) is called a *discrete trajectory*; the corresponding collection

$$\Big\{\frac{x_N(t_{j+1}) - x_N(t_j)}{h_N} \,\Big|\, j = 0, \ldots, k\Big\}$$

is called a *discrete velocity*. Extended discrete velocities to $[a, b]$, denoted by $\dot{x}_N(t)$, are defined as piecewise constant extensions of discrete velocities for $t \in [t_j, t_{j+1})$, $j = 0, \ldots, k$. The extended discrete trajectories to $[a - \Delta, b]$, denoted by $x_N(t)$, are defined as piecewise constant extensions of discrete trajectories on $[a - \Delta, a)$ and piecewise linear extensions on $[a, b]$. It is easy to see that

$$x_N(t) = \bar{x}(a) + \int_a^t \dot{x}_N(s)\,ds, \quad t \in [a, b].$$

The next two theorems (see Wang [19]) justify the *strong approximation* of continuous-time trajectories for the neutral inclusion by discrete ones as well as the *strong convergence* of discrete optimal solutions; cf. also [10, 15].

Theorem 2.1. *Assume that $\bar{x}(t)$ is a trajectory for* (1.2) *under hypotheses (H1), (H2), and (H3). Then there is a sequence of solutions to* (2.1)*, $z_N(t_j)$, $j = -N, \ldots, k+1$, such that $z_N(t_0) = \bar{x}(a)$, the extended discrete trajectories $z_N(t)$ converge to $\bar{x}(t)$ uniformly on $[a - \Delta, b]$, and the extended discrete velocities $\dot{z}_N(t)$ converge to $\dot{\bar{x}}(t)$ in the L^2-norm on $[a, b]$ as $N \to \infty$.*

In what follows we always assume that $\bar{x}(t)$, $a - \Delta \le t \le b$, is an optimal solution to (P). Employing Theorem 2.1, construct the sequence of discrete-time optimization problems (P_N), $N \in I\!N$, defined as follows:

$$\begin{aligned} \text{minimize} \quad & J_N[x_N] := \varphi(x_N(a), x_N(b)) + |x_N(a) - \bar{x}(a)|^2 \\ & + h_N \sum_{j=0}^{k} f(x_N(t_j), x_N(t_{j-N}), t_j) \\ + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} & \Big| \frac{z_N(t_{j+1}) - A z_N(t_{j+1-N}) - z_N(t_j) + A z_N(t_{j-N})}{h_N} \\ & - [\dot{\bar{x}}(t) - A\dot{\bar{x}}(t - \Delta)] \Big|^2 dt, \end{aligned} \tag{2.2}$$

subject to the dynamic and endpoint constraints:

$$\begin{aligned} x_N(t_{j+1}) - A x_N(t_{j+1-N}) \in x_N(t_j) - A x_N(t_{j-N}) \\ + h_N F(x_N(t_j), x_N(t_{j-N}), t_j), \quad j = 0, \ldots, k, \end{aligned} \tag{2.3}$$

$$x_N(t_j) = c(t_j), \quad j = -N, \ldots, -1, \tag{2.4}$$

$$(x_N(a), x_N(b)) \in \Omega_N := \Omega + \eta_N B, \tag{2.5}$$

$$|x_N(t_j) - \bar{x}(t_j)| \le \epsilon, \quad j = 0, \ldots, k+1, \tag{2.6}$$

where $\epsilon > 0$ is a fixed number, and where $\eta_N := |z_N(b) - \bar{x}(b)|$ is constructively built in Theorem 2.1.

To obtain necessary optimality conditions for (P), we need to impose an intrinsic property of (P) called *relaxation stability*. This property means that the optimal value (infimum) of the cost functional in the original problem agrees with the one in its relaxation/convexification; see [6],[10],[15], and [20] for more details, discussions, and efficient conditions ensuring the relaxation stability in various settings. Note that the relaxation stability of (P) always holds if the velocity sets $F(x,y,t)$ are convex.

The next strong convergence theorem builds a bridge between optimization problems for neutral functional-differential inclusions and their discrete-time counterparts; it is one of the basic ingredients for deriving necessary optimality conditions in (P) by the limiting process via discrete approximations.

Theorem 2.2. *Suppose that assumption (H1)–(H4) are satisfied and that problem (P) is stable with respect to relaxation. Then for any sequence of optimal solutions $\bar{x}_N(t_j)$, $j=-N,\ldots,k+1$, to (P_N) the extended trajectories $\bar{x}_N(t)$ converge uniformly to $\bar{x}(t)$ on $[a-\Delta,b]$ while the extended velocities $\dot{\bar{x}}_N(t)$ converge to $\dot{\bar{x}}(t)$ in the L^2-norm on $[a,b]$ as $N\to\infty$.*

3. Tools of Generalized Differentiation

This section describes generalized differential tools of variational analysis for nonsmooth and set-valued objects that used in the paper to derive necessary optimality conditions for discrete-time and continuous-time inclusions. We refer the reader to [8, 10] and [17] for details and discussions.

Recall that the *basic/limiting normal cone* to the set $\Omega\subset I\!\!R^n$ at the point $\bar{x}\in\Omega$ is defined by

$$N(\bar{x};\Omega):=\mathop{\mathrm{Lim\,sup}}_{x\xrightarrow{\Omega}\bar{x}}\widehat{N}(x;\Omega), \tag{3.1}$$

where $x\xrightarrow{\Omega}\bar{x}$ means that $x\to\bar{x}$ with $x\in\Omega$, and where

$$\widehat{N}(\bar{x};\Omega):=\Big\{x^*\in I\!\!R^n\Big|\limsup_{x\xrightarrow{\Omega}\bar{x}}\frac{\langle x^*,x-\bar{x}\rangle}{|x-\bar{x}|}\le 0\Big\} \tag{3.2}$$

is the so-called cone of Fréchet normals to Ω at $\bar{x}$. For convex sets Ω both cones $N(\bar{x};\Omega)$ and $\widehat{N}(\bar{x};\Omega)$ reduce to the normal cone of convex analysis. Note also that the basic normal cone (3.1) is often nonconvex while satisfying a comprehensive calculus in contrast to (3.2).

Given an extended real-valued function $\varphi\colon I\!\!R^n\to\overline{I\!\!R}:=[-\infty,\infty]$ finite at $\bar{x}$, the *basic subdifferential* of φ at $\bar{x}$ is

$$\partial\varphi(\bar{x}):=\big\{x^*\in I\!\!R^n\big|\,(x^*,-1)\in N((\bar{x},\varphi(\bar{x}));\mathrm{epi}\,\varphi)\big\}, \tag{3.3}$$

where $\mathrm{epi}\,\varphi:=\{(x,\mu)\in I\!\!R^{n+1}|\,\mu\ge\varphi(x)\}$.

The *coderivative* $D^*F(\bar{x},\bar{y})\colon\ I\!\!R^m\rightrightarrows I\!\!R^n$ of a set-valued mapping $F\colon I\!\!R^n\rightrightarrows I\!\!R^m$ at a point $(\bar{x},\bar{y})\in\mathrm{gph}\,F$ is defined by

$$D^*F(\bar{x},\bar{y})(y^*):=\big\{x^*\in I\!\!R^n\big|\,(x^*,-y^*)\in N((\bar{x},\bar{y});\mathrm{gph}\,F)\big\}. \tag{3.4}$$

In this paper we need the following properties of generalized differentiation.

Proposition 3.1. *Suppose that f is locally Lipschitzian around $\bar{x}$ with modulus l_f. Then one has*

$$\partial f(\bar{x}) \neq \emptyset \quad \textit{and} \quad |x^*| \le l_f,\ x^* \in \partial f(\bar{x}).$$

Proposition 3.2. *Let f_1 and f_2 be two lower semicontinuous functions one of which is locally Lipschitzian around $\bar{x}$. Then*

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Proposition 3.3. *Let $F : I\!\!R^n \rightrightarrows I\!\!R^m$ be a closed-graph set-valued mapping locally bounded around $\bar{x}$. Then the following properties are equivalent:*

1. *F is locally Lipschitzian at $\bar{x}$.*
2. *There exist a neighborhood U of $\bar{x}$ and a number $L > 0$ such that*

$$\sup\{|x^*| : x^* \in D^*F(x,y)(y^*)\} \le L|y^*| \quad \textit{for}\ x \in U,\ y \in F(x),\ y^* \in I\!\!R^m.$$

Note that the latter *coderivative criterion* for Lipschitzian stability established in [9] (see also [17, Theorem 9.40]) plays a crucial role in justifying the required convergence of *adjoint* trajectories in discrete approximations; see Section 5.

For applications in this paper we also need the following extensions of the basic constructions (3.1), (3.3), and (3.4) to the case of sets, functions, and set-valued mappings depending on parameters; cf. [11, 13].

Given a moving set $\Omega\colon [a,b] \rightrightarrows I\!\!R^n$ and $\bar{x} \in \Omega(\bar{t})$, the *extended normal cone* to $\Omega(\bar{t})$ at $\bar{x}$ is defined by

$$\widetilde{N}(\bar{x};\Omega(\bar{t})) := \mathop{\mathrm{Limsup}}_{(t,x)\to(\bar{t},\bar{x}),\ (t,x)\in \mathrm{gph}\,\Omega} \widehat{N}(x;\Omega(t)). \tag{3.5}$$

If $\varphi\colon I\!\!R^n \times [a,b] \to \overline{I\!\!R}$ is finite at $(\bar{x},\bar{t})$, the *extended subdifferential* of φ at $(\bar{x},\bar{t})$ with respect to x is defined by

$$\widetilde{\partial}_x \varphi(\bar{x},\bar{t}) := \big\{x^* \in I\!\!R^n \big|\ (x^*,-1) \in \widetilde{N}((\bar{x},\varphi(\bar{x},\bar{t}));\mathrm{epi}\,\varphi(\cdot,\bar{t}))\big\}. \tag{3.6}$$

Given $F\colon I\!\!R^n \times [a,b] \rightrightarrows I\!\!R^m$ and $\bar{y} \in F(\bar{x},\bar{t})$, the *extended coderivative* of F at $(\bar{x},\bar{y},\bar{t}) \in \mathrm{gph}\,F$ with respect to x is defined by

$$\widetilde{D}^*_x F(\bar{x},\bar{y},\bar{t})(y^*) := \big\{x^* \in I\!\!R^n \big|\ (x^*,-y^*) \in \widetilde{N}((\bar{x},\bar{y});\mathrm{gph}\,F(\cdot,\bar{t}))\big\}, \tag{3.7}$$

where $y^* \in I\!\!R^m$. Note that the sets (3.5)–(3.7) may be bigger in some situations than the corresponding sets $N(\bar{x};\Omega(\bar{t}))$, $\partial_x\varphi(\bar{x},\bar{t})$, and $D^*_x F(\bar{x},\bar{y},\bar{t})(y^*)$, where the latter two sets stand for the subdifferential (3.3) of $\varphi(\cdot,\bar{t})$ at $\bar{x}$ and the coderivative (3.4) of $F(\cdot,\bar{t})$ at $(\bar{x},\bar{y},\bar{t})$, respectively. Efficient conditions ensuring equalities for these sets are discussed in [11], [13], [14], and [15]. In particular, the following *robustness property* holds.

Proposition 3.4. *Let $\Omega\colon [a,b] \rightrightarrows I\!\!R^n$ with $\bar{x} \in \Omega(\bar{t})$. Then*

$$\widetilde{N}(\bar{x};\Omega(\bar{t})) = \operatorname*{Limsup}_{(t,x)\to(\bar{t},\bar{x}),\ (t,x)\in \mathrm{gph}\Omega} \widetilde{N}(x;\Omega(t)). \tag{3.8}$$

Finally in this section, consider the nonsmooth problem (MP) of mathematical programming with *many geometric constraints* given by:

$$\begin{cases} \text{minimize } \phi_0(z) \quad \text{subject to} \\ \phi_j(z) \le 0, \quad j = 1,\ldots,r, \\ g_j(z) = 0, \quad j = 0,\ldots,m, \\ z \in \Lambda_j, \quad j = 0,\ldots,l, \end{cases}$$

where $\phi_j\colon I\!\!R^d \to I\!\!R$, $g_j\colon I\!\!R^d \to I\!\!R^n$, and $\Lambda_j \subset I\!\!R^d$. We need the following version of the generalized Lagrange multiplier rule established in [8, Corollary 7.5.1].

Theorem 3.5. *Let $\bar{z}$ be an optimal solution to (MP). Assume that all ϕ_i are Lipschitz continuous, that g_j are continuously differentiable, and that Λ_j are locally closed near $\bar{z}$. Then there exist real numbers $\{\mu_j|\ j = 0,\ldots,r\}$ as well as vectors $\{\psi_j \in I\!\!R^n|\ j = 0,\ldots,m\}$ and $\{z_j^* \in I\!\!R^d|\ j = 0,\ldots,l\}$, not all zero, such that*

$$\mu_j \ge 0, \quad j = 0,\ldots,r, \tag{3.9}$$

$$\mu_j\phi_j(\bar{z}) = 0, \quad j = 1,\ldots,r, \tag{3.10}$$

$$z_j^* \in N(\bar{z};\Lambda_j), \quad j = 0,\ldots,l, \tag{3.11}$$

$$-\sum_{j=0}^{l} z_j^* \in \partial\Big(\sum_{j=0}^{r}\mu_j\phi_j\Big)(\bar{z}) + \sum_{j=0}^{m}\nabla g_j(\bar{z})^*\psi_j. \tag{3.12}$$

4. Necessary Optimality Conditions for Discrete Approximations

In this section we reduce the discrete-time dynamic optimization problem (P_N), for each $N \in I\!\!N$, to the mathematical programming problem (MP) with many geometric constraints considered in Section 3. Then applying Theorem 3.5 to (MP), we derive in this way necessary optimality conditions for discrete approximation problems (P_N) with the use of generalized differential calculus.

Let $z^N := (x_0^N, x_1^N, \ldots, x_{k+1}^N, y_0^N, y_1^N, \ldots, y_k^N) \in I\!\!R^{n(2k+3)}$. Define

$$\begin{aligned} \phi_0(z) := \varphi(x_0^N, x_{k+1}^N) + |x_0^N - \bar{x}(a)|^2 + h_N\sum_{j=0}^{k} f\big(x_j^N, x_{j-N}^N, t_j\big) \\ + \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} \big|y_j^N - [\dot{\bar{x}}(t) - A\dot{\bar{x}}(t-\Delta)]\big|^2\,dt, \end{aligned} \tag{4.1}$$

$$\begin{gathered}\phi_j(z) := |x_j^N - \bar{x}(t_j)| - \varepsilon, \quad j = 1, \ldots, k+1, \\ \Lambda_j := \big\{(x_0^N, \ldots, y_k^N) \;\big|\; y_j^N \in F(x_j^N, x_{j-N}^N, t_j)\big\}, \quad j = 0, \ldots, k, \\ \Lambda_{k+1} := \big\{(x_0^N, \ldots, y_k^N) \;\big|\; (x_0^N, x_{k+1}^N) \in \Omega_N\big\}, \\ g_j(z) := x_{j+1}^N - Ax_{j+1-N}^N - x_j^N + Ax_{j-N}^N - h_N y_j^N, \quad j = 0, \ldots, k,\end{gathered}$$

where $x_j^N := c(t_j)$ for $j < 0$. Let $\bar{z}^N = (\bar{x}_0^N, \ldots, \bar{x}_{k+1}^N, \bar{y}_0^N, \ldots, \bar{y}_k^N)$ be an optimal solution to (MP). Applying Theorem 3.5, we find real numbers μ_j^N and vectors $z_j^* \in I\!R^{n(2k+3)}$ for $j = 0, \ldots, k+1$ as well as vectors $\psi_j^N \in I\!R^n$ for $j = 0, \ldots, k$, not all zero, such that conditions (3.9)–(3.12) are satisfied.

Taking $z_j^* = (x_{0,j}^*, \ldots, x_{k+1,j}^*, y_{0,j}^*, \ldots, y_{k,j}^*) \in N(\bar{z}^N; \Lambda_j)$ for $j = 0, \ldots, k$, we observe that all but one components of z_j^* are zero and the remaining one satisfies

$$(x_{j,j}^*, x_{j-N,j}^*, y_{j,j}^*) \in N((\bar{x}_j^N, \bar{x}_{j-N}^N, \bar{y}_j^N); \operatorname{gph} F(\cdot, \cdot, t_j)), \quad j = 0, \ldots, k.$$

Similarly, the condition $z_{k+1}^* \in N(\bar{z}^N; \Lambda_{k+1})$ is equivalent to

$$(x_{0,k+1}^*, x_{k+1,k+1}^*) \in N((\bar{x}_0^N, \bar{x}_{k+1}^N); \Omega_N)$$

with all the other components of z_{k+1}^* equal to zero. By Theorem 2.2 we conclude that $\phi_j(\bar{z}^N) < 0$ for $j = 1, \ldots, k+1$ whenever N is sufficiently large. Thus $\mu_j^N = 0$ for these indexes due to the complementary slackness conditions (3.10). Let $\lambda^N := \mu_0^N \geq 0$. Observe further that

$$\begin{gathered}\sum_{j=0}^{k} (\nabla g_j(\bar{z}^N))^* \psi_j^N \\ = \big(-\psi_0 + A^*(\psi_N^N - \psi_{N-1}^N),\ \psi_0 - \psi_1 + A^*(\psi_{N+1}^N - \psi_N^N),\ \cdots, \\ \psi_{k-N-1} - \psi_{k-N} + A^*(\psi_k^N - \psi_{k-1}^N), \psi_{k-N} - \psi_{k-N+1} - A^*\psi_k^N, \cdots, \\ \psi_{k-1}^N - \psi_k^N, \psi_k^N,\ -h_N\psi_0^N, \cdots, -h_N\psi_k^N\big).\end{gathered}$$

From the subdifferential sum rule of Proposition 3.2 applied to ϕ_0 in (4.1) we have

$$\begin{aligned}\partial\phi_0(\bar{z}^N) \subset \partial\varphi(\bar{x}_0^N, \bar{x}_{k+1}^N) + 2(\bar{x}_0^N - \bar{x}(a)) + h_N \sum_{j=0}^{k} \partial f\big(\bar{x}_j^N, \bar{x}_{j-N}^N, t_j\big) \\ + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} 2\big(\bar{y}_j^N - [\dot{\bar{x}}(t) - A\dot{\bar{x}}(t-\Delta)]\big)\, dt,\end{aligned}$$

where ∂f is the basic subdifferential of f with respect to the first two variables.

Thus inclusion (3.12) in Theorem 3.5 implies that

$$
\begin{gathered}
-x^*_{0,0} - x^*_{0,N} - x^*_{0,k+1} = \lambda^N u^N_0 + \lambda^N h_N \vartheta^N_0 + \lambda^N h_N \kappa^N_0 \\
+2\lambda^N(\bar{x}^N_0 - \bar{x}(a)) - \psi^N_0 - A^*(\psi^N_{N-1} - \psi^N_N), \\
-x^*_{j,j} - x^*_{j,j+N} = \lambda^N h_N \vartheta^N_j + \lambda^N h_N \kappa^N_j + \psi^N_{j-1} - \psi^N_j \\
-A^*(\psi^N_{j+N-1} - \psi^N_{j+N}), \quad j = 1, \ldots, k-N, \\
-x^*_{k-N+1,k-N+1} = \lambda^N h_N \vartheta^N_{k-N+1} + \psi^N_{k-N} - \psi^N_{k-N+1} + A^*\psi^N_k, \\
-x^*_{j,j} = \lambda^N h_N \vartheta^N_j + \psi^N_{j-1} - \psi^N_j, \quad j = k-N+2, \ldots, k, \\
-x^*_{k+1,k+1} = \lambda^N u^N_{k+1} + \psi^N_k, \\
-y^*_{j,j} = \lambda^N \theta^N_j - h_N \psi^N_j, \quad j = 0, \ldots, k,
\end{gathered}
$$

with the notation

$$
\begin{gathered}
(u^N_0, u^N_{k+1}) \in \partial\varphi(\bar{x}^N_0, \bar{x}^N_{k+1}), \quad (\vartheta^N_j, \kappa^N_{j-N}) \in \partial f(\bar{x}^N_j, \bar{x}^N_{j-N}, t_j), \\
\theta^N_j := -2 \int_{t_j}^{t_{j+1}} \left(\dot{\bar{x}}(t) - A\dot{\bar{x}}(t-\Delta) - \bar{y}^N_j\right) dt.
\end{gathered}
$$

Let $f_j(\cdot,\cdot) := f(\cdot,\cdot,t_j)$ and $F_j(\cdot,\cdot) := F(\cdot,\cdot,t_j)$. Based on the above relationships, we derive the following necessary optimality conditions for discrete-time problems (P_N) governed by neutral functional-difference control systems.

Theorem 4.1. *Let $\bar{z}^N$ be an optimal solution to problem (P_N). Assume that* $\operatorname{gph} F_j$ *is closed for each $j = 1, \ldots, k$, and that the functions φ is Lipschitz continuous around the point $(\bar{x}^N_0, \bar{x}^N_{k+1})$. Then there exist $\lambda^N \geq 0$, p^N_j $(j = 0, \ldots, k+N+1)$, and q^N_j $(j = -N, \ldots, k+1)$, not all zero, such that*

$$
\begin{gathered}
\Big(\frac{P^N_{j+1} - P^N_j}{h_N} - q^N_j,\ q^N_{j-N},\ -\frac{\lambda^N \theta^N_j}{h_N} + p^N_{j+1}\Big) \\
\in \lambda^N(\vartheta^N_j, \kappa^N_{j-N}, 0) + N((\bar{x}^N_j, \bar{x}^N_{j-N}, \bar{y}^N_j);\ \operatorname{gph} F_j), \quad j = 1, \ldots, k,
\end{gathered} \tag{4.2}
$$

$$
p^N_j = 0,\ \ j = k+2, \ldots, k+N+1, \tag{4.3}
$$

$$
q^N_j = 0,\ \ j = k-N+1, \ldots, k+1, \tag{4.4}
$$

$$
(p^N_0, -p^N_{k+1}) \in \lambda^N \partial\varphi(\bar{x}^N_0, \bar{x}^N_{k+1}) + N((\bar{x}^N_0, \bar{x}^N_{k+1}); \Omega_N), \tag{4.5}
$$

with the notation

$$
P^N_j = p^N_j - A^* p^N_{j+N}.
$$

Proof. Let $p^N_j := \psi^N_{j-1}$ for $j = 1, \ldots, k+1$, and $p^N_j := 0$ for $j = k+2, \ldots, k+N+1$, let $q^N_j := \lambda^N \kappa^N_j + x^*_{j,j+N}/h_N$ for $j = -N, \ldots, k-N$, and $q^N_j := 0$ for $j = k-N+1, \ldots, k+1$, and let $p^N_0 := \lambda^N u^N_0 + x^*_{0,k+1}$. Then it is easy to verify that all the relationships (4.2)–(4.5) hold. □

Corollary 4.2. *In addition to the assumptions of Theorem 4.1 suppose that F_j is bounded and Lipschitz continuous around $(\bar{x}^N_j, \bar{x}^N_{j-N})$ for each $j = 0, \ldots, k$. Then* *conditions* (4.2)–(4.5) *and* $\lambda^N \geq 0$ *hold with* $(\lambda^N, p^N_{k+1}) \neq 0$ *simultaneously; i.e., we can set*

$$(\lambda^N)^2 + |p^N_{k+1}|^2 = 1. \tag{4.6}$$

Proof. The proof is similar to the one for Corollary 5.2 in [15]. □

5. Necessary Optimality Conditions for Functional-Differential Inclusions

The main results of this paper are given in this section. By passing to the limit in discrete necessary optimality conditions obtained in Section 4, we derive necessary optimality conditions for problem (P) in both Euler-Lagrange and Hamiltonian forms involving generalized differential constructions of Section 3. Note that, in contrast to neutral systems in the Hale form in [15], we ensure the *absolute continuity* of the adjoint arcs $p(\cdot)$ and $q(\cdot)$ in the main theorem on the corresponding intervals.

Theorem 5.1. *Let* $\bar{x}(\cdot)$ *be an optimal solution to problem* (P) *under assumptions (H1)–(H4). Suppose also that* (P) *is stable with respect to relaxation. Then there exist a nonnegative number* λ *and two absolutely continuous adjoint arcs* $p\colon [a, b+\Delta] \to I\!R^n$ *and* $q\colon [a-\Delta, b] \to I\!R^n$ *such that the following conditions hold:*

$$\lambda_0 + |p(b)| = 1, \tag{5.1}$$

$$p(t) = 0, \quad t \in (b, b+\Delta], \tag{5.2}$$

$$q(t) = 0, \quad t \in (b-\Delta, b], \tag{5.3}$$

$$(p(a)+q(a), -p(b)) \in \lambda\partial\varphi(\bar{x}(a), \bar{x}(b)) + N((\bar{x}(a), \bar{x}(b)); \Omega), \tag{5.4}$$

$$\begin{gathered}\big(\dot{p}(t) - A^*\dot{p}(t+\Delta),\ \dot{q}(t-\Delta) - A^*\dot{q}(t)\big) \\ \in \mathrm{co}\big\{(u, w, p(t)+q(t)) \in \lambda(\widetilde{\partial} f(\bar{x}(t), \bar{x}(t-\Delta), t), 0) \\ +\widetilde{N}((\bar{x}(t), \bar{x}(t-\Delta), \dot{\bar{x}}(t) - A\dot{\bar{x}}(t-\Delta)); \mathrm{gph}\, F(t))\big\}\ \text{a.e. } t \in [a,b].\end{gathered} \tag{5.5}$$

Proof. Employing Theorem 2.2, construct a sequence of optimal solutions $\bar{x}^N = (\bar{x}^N_0, \ldots, \bar{x}^N_{k+1})$ to (P_N) that strongly approximates to the optimal solution $\bar{x}(t)$ to (P). Then Theorem 4.1 applied to $\bar{x}^N$ ensures the existence of a real number $\lambda^N \geq 0$ as well as vectors p^N_j $(j = 0, \ldots, k+N+1)$ and q^N_j $(j = -N, \ldots, k+1)$, not all zero, such that (4.2)–(4.5) are satisfied.

Due to Corollary 4.2 one has that $\lambda^N \to \lambda \geq 0$ along some subsequence of $N \to \infty$. In what follows we use the notation $p^N(t)$ and $\bar{x}^N(t)$, $q^N(t)$ for piecewise linear extensions of the corresponding discrete mappings on $[a, b+\Delta]$ and $[a-\Delta, b]$, respectively.

Let $\theta^N(t) := \theta^N_j / h_N$ for $t \in [t_j, t_{j+1})$ and $j = 0, 1, \ldots, k$. Theorem 2.2 gives

$$\begin{aligned}\int_a^b |\theta^N(t)|\, dt &= \sum_{j=0}^k |\theta^N_j| \leq 2\sum_{j=0}^k \int_{t_j}^{t_{j+1}} \big|\dot{\bar{x}}(t) - A\dot{\bar{x}}(t-\Delta) - \bar{y}^N_j\big|\, dt \\ &= 2\int_a^b \big|\dot{\bar{x}}(t) - A\dot{\bar{x}}(t-\Delta) - [\dot{\bar{x}}^N(t) - A\dot{\bar{x}}^N(t-\Delta)]\big|\, dt := \nu_N \to 0.\end{aligned}$$

Without loss of generality suppose that

$$\dot{\bar{x}}^N(t) - A\dot{\bar{x}}^N(t-\Delta) \to \dot{\bar{x}}(t) - A\dot{\bar{x}}(t-\Delta) \quad \text{and} \quad \theta^N(t) \to 0 \ \text{ as } N \to \infty \tag{5.6}$$

for a.e. $t \in [a,b]$. Let us now establish the uniform boundedness of $(p^N(t), q^N(t-\Delta))$ for sufficiently large N based on the above coderivative characterization of Lipschitzian multifunctions. Indeed, taking into account that $p^N_{j+N} = 0$ and $q^N_j = 0$ for $j = k-N+2, \ldots, k+1$, we get from (4.2) that

$$\Big(\frac{p^N_{j+1} - p^N_j}{h_N} - \lambda^N \vartheta^N_j,\, q^N_{j-N} - \lambda^N \kappa^N_{j-N},\, -\frac{\lambda^N \theta^N_j}{h_N} + p^N_{j+1}\Big) \in N((\bar{x}^N_j, \bar{x}^N_{j-N}, \bar{y}^N_j); \mathrm{gph} F_j).$$

By the definition of coderivative (3.4) one has

$$\Big(\frac{p^N_{j+1} - p^N_j}{h_N} - \lambda^N \vartheta^N_j,\, q^N_{j-N} - \lambda^N \kappa^N_{j-N}\Big) \in D^* F_j(\bar{x}^N_j, \bar{x}^N_{j-N}, \bar{y}^N_j)\Big(\frac{\lambda^N \theta^N_j}{h_N} - p^N_{j+1}\Big).$$

Then Proposition 3.3 yields that

$$\Big|\Big(\frac{p^N_{j+1} - p^N_j}{h_N} - \lambda^N \vartheta^N_j,\, q^N_{j-N} - \lambda^N \kappa^N_{j-N}\Big)\Big| \le L_F \Big|\frac{\lambda^N \theta^N_j}{h_N} - p^N_{j+1}\Big|. \tag{5.7}$$

Since $|(v^N_j, \kappa^N_{j-N})| \le l_f$ due to the Proposition 3.1 and the Lipschitz continuity of f with modulus l_f, one further has

$$|p^N_{j+1} - p^N_j| \le |(p^N_{j+1} - p^N_j, h_N q^N_{j-N})| \le L_F |\theta^N_j| + l_f h_N + L_F h_N |p^N_{j+1}|$$

that together with (4.6) ensure the estimate

$$\begin{aligned} |p^N_j| &\le L_F |\theta^N_j| + l_f h_N + (L_F h_N + 1)|p^N_{j+1}| \\ &\le L_F |\theta^N_j| + (L_F h_N + 1) L_F |\theta^N_{j+1}| + l_f h_N + (L_F h_N + 1) l_f h_N \\ &+ (L_F h_N + 1)^2 |p^N_{j+2}| \le \cdots \le \exp[L_F(b-a)](1 + l_f(b-a) + L_F \nu_N). \end{aligned}$$

Thus the sequence $\{p^N_j \mid j = k-N+2, \ldots, k+1\}$ is uniformly bounded. Furthermore, (5.7) implies

$$|q^N_{j-N} - \lambda^N \kappa^N_{j-N}| \le L_F |\theta^N_j| / h_N + L_F |p^N_{j+1}|,$$

which justifies the uniform boundedness of $\{q^N_{j-N} \mid j = k-N+2, \ldots, k+1\}$ follows. Therefore the sequence $\{p^N(t), q^N(t-\Delta)\}$ is uniformly bounded on $[b-\Delta, b]$ and so is $\{P^N(t), q^N(t-\Delta)\}$ on $[b-\Delta, b]$.

Considering discrete inclusions (4.2) for $j = k-2N+2, \ldots, k-N+1$, we get

$$\begin{aligned} &\Big|\Big(\frac{p^N_{j+1} - p^N_j}{h_N} - \lambda^N \vartheta^N_j,\ q^N_{j-N} - \lambda^N \kappa^N_{j-N}\Big)\Big| \\ &\le L_F \Big|\frac{\lambda^N \theta^N_j}{h_N} - p^N_{j+1}\Big| + \Big|\Big(\frac{A^* p^N_{j+N+1} - A^* p^N_{j+N}}{h_N} - q^N_j,\, 0\Big)\Big|. \end{aligned}$$

This readily implies the estimates

$$|p_{j+1}^N - p_j^N| \le L_F|\theta_j^N| + l_f h_N + L_F h_N |p_{j+1}^N| + L_q h_N \\ +|A|\cdot|p_{j+N}^N| + |A|\cdot|p_{j+N+1}^N| \ \text{ and}$$

$$|p_j^N| \le L_F|\theta_j^N| + (l_f + L_q)h_N + (L_F h_N + 1)|p_{j+1}^N| \\ +|A|\cdot|p_{j+N}^N| + |A|\cdot|p_{j+N+1}^N|,$$

where L_q stands for a uniform bound of $\{q_j^N\}$. As before, the latter estimates ensure the uniform boundedness of p_j^N and q_{j-N}^N for $j = k-2N+2, \cdots, k-N+1$, and thus the boundedness of the sequences $\{p^N(t), q^N(t-\Delta)\}$ and $\{P^N(t), q^N(t-\Delta)\}$ on the interval $[b-2\Delta, b-\Delta]$. Repeating this procedure, we justify the boundedness of $\{p^N(t)\}$, $\{P^N(t)\}$, and $\{q^N(t)\}$ on the whole intervals $[a, b+\Delta]$, $[a,b]$, and $[a-\Delta, b]$.

Next let us estimate $\dot P^N(t)$. For $t_j \le t < t_{j+1}$ with $j = 0, \ldots, k$ we have

$$|\dot P^N(t)| \le |\dot P^N(t) - q_j^N| + |q_j^N| \le \Big|\Big(\frac{P_{j+1}^N - P_j^N}{h_N} - q_j^N, q_{j-N}^N\Big)\Big| + |q_j^N| \\ \le L_F\Big|\frac{\lambda^N\theta_j^N}{h_N} - p_{j+1}^N\Big| + l_f + |q_j^N|,$$

which gives the uniform boundedness of $\dot P^N(t)$ on $[a,b]$. Hence the sequence $\{\dot P^N(\cdot)\}$ is weakly compact in $L^1[a,b]$. Thus there is an absolutely continuous mapping $P\colon [a,b] \to I\!R^n$ such that $\dot P^N(t) \to \dot P(t)$ weakly in $L^1[a,b]$ and that $P^N(t) \to P(t)$ uniformly on $[a,b]$ as $N \to \infty$. Since $p^N(t)$ and $q^N(t-\Delta)$ are uniformly bounded on $[a, b+\Delta]$, they surely converge to some mappings $\widetilde p(t)$ and $\widetilde q(t-\Delta)$ weakly in $L^2[a, b+\Delta]$. Thus we have $P(t) = \widetilde p(t) - A^*\widetilde p(t+\Delta)$ for $t \in [a,b]$.

Rewrite now (4.2) in the form

$$\begin{aligned}(\dot P^N(t) - q^N(t), q^N(t-\Delta)) \in \big\{(u,v) \mid \big(u, v, p^N(t_{j+1})\big) - \lambda^N\theta_j^N/h_N\big) \\ \in \lambda^N(\partial f(\bar x(t_j), \bar x(t_j-\Delta), t_j), 0) \\ +N(\bar x_j^N, \bar x_{j-N}^N, \bar y_j^N); \mathrm{gph} F_j)\big\},\ t \in [t_j, t_{j+1}],\ j = 0,1,\ldots,k.\end{aligned} \tag{5.8}$$

According to the classical Mazur theorem, there is a sequence of convex combinations of $(\dot P^N(t) - q^N(t),\ q^N(t-\Delta))$ that converges to $(\dot P(t) - \widetilde q(t),\ \widetilde q(t-\Delta))$ for a.e. $t \in [a,b]$. Passing to the limit in (5.8) as $N \to \infty$ and using (5.6), we obtain the inclusion

$$\begin{aligned}\big(\dot P(t) - \widetilde q(t),\ \widetilde q(t-\Delta)\big) \in \mathrm{co}\big\{(u, w, \widetilde p(t)) \in \lambda(\widetilde\partial f(\bar x(t), \bar x(t-\Delta), t), 0) \\ +\widetilde N((\bar x(t), \bar x(t-\Delta), \dot{\bar x}(t) - A\dot{\bar x}(t-\Delta)); \mathrm{gph}\, F(t))\big\} \ \text{ a.e. } \ t \in [a,b]\end{aligned} \tag{5.9}$$

with the normalization condition

$$\lambda + |\widetilde p(b)| = 1 \tag{5.10}$$

that follows from (4.6). Furthermore, (4.3) and (4.4) give

$$\widetilde p(t) = 0, \quad t \in (b, b+\Delta], \tag{5.11}$$

$$\widetilde q(t) = 0, \quad t \in (b-\Delta, b]. \tag{5.12}$$

By the robustness property of the basic subdifferential one has

$$\lambda^N \partial\varphi(\bar{x}_0^N, \bar{x}_{k+1}^N) \to \lambda\partial\varphi(\bar{x}(a), \bar{x}(b)).$$

Taking this into account and also that $\Omega_N = \Omega + \eta_N I\!B$, we get

$$(\widetilde{p}(a), -\widetilde{p}(b)) \in \lambda\partial\varphi(\bar{x}(a), \bar{x}(b)) + N((\bar{x}(a), \bar{x}(b)); \Omega). \tag{5.13}$$

by passing to the limit in (4.5). Define $q : [a - \Delta, b] \to I\!R^n$ by

$$\widetilde{q}(t) := \dot{q}(t) - A^*(t + \Delta)\dot{q}(t + \Delta)$$

and put $p(t) := \widetilde{p}(t) - q(t)$. Then $q(t) = 0$ on $(b - \Delta, b]$ while both $q(\cdot)$ and $q(\cdot)$ are absolutely continuous on the corresponding intervals. Finally, it is easy to verify that the required relations (5.1)–(5.5) follow from the obtained relations (5.9)–(5.13). This completes the proof of the theorem. □

In conclusion of this section, consider a special case of problem (P) with $f = 0$ called the *Mayer problem* for neutral functional-differential inclusions and labelled as (M). Let us prove that the *extended Euler-Lagrange inclusion* (5.5) obtained in Theorem 5.1 implies two other principal necessary optimality conditions expressed in terms of the Hamiltonian function built upon the mapping F in (1.2). The first condition called the *extended Hamiltonian inclusion* is given below in terms of a *partial convexification* of the basic subdifferential (3.3) for the Hamiltonian function. The second one is an analog of the classical *Weierstrass-Pontryagin maximum condition* for neutral functional-differential inclusions.

Define the *Hamiltonian function* for system (1.2) assumed for simplicity to be autonomous by

$$H(x, y, p) := \max\big\{\langle p, v\rangle \;\big|\; v \in F(x, y)\big\}$$

and consider the basic subdifferential ∂H of H with respect to (x, y).

Theorem 5.2. *Let $\bar{x}(\cdot)$ be an optimal solution to problem (M) under the assumptions made. Suppose also that (M) is stable with respect to relaxation. Then there exist a nonnegative number λ and two* absolutely continuous *adjoint arcs $p\colon [a, b+\Delta] \to I\!R^n$ and $q\colon [a-\Delta, b] \to I\!R^n$ such that, besides the necessary conditions of Theorem 5.1, one has the maximum condition*

$$\big\langle p(t) + q(t), \dot{\bar{x}}(t) - A\dot{\bar{x}}(t - \Delta)\big\rangle = H(\bar{x}(t), \bar{x}(t - \Delta), p(t) + q(t)) \tag{5.14}$$

and the Hamiltonian inclusion

$$\begin{aligned} \big(\dot{p}(t) - A^*\dot{p}(t + \Delta),\, \dot{q}(t - \Delta) - A^*\dot{q}(t)\big) \in \operatorname{co}\big\{(u, w) \,|\, (-u, -w,\\ \dot{\bar{x}}(t) - A\dot{\bar{x}}(t - \Delta)) \in \partial H(\bar{x}(t), \bar{x}(t - \Delta), p(t) + q(t))\big\} \end{aligned} \tag{5.15}$$

for a.e. $t \in [a, b]$. Moreover, if F is convex-valued around $(\bar{x}(t), \bar{x}(t - \Delta))$, then the extended Hamiltonian inclusion (5.15) *is equivalent to the extended Euler-Lagrange inclusion in the coderivative form*

$$\begin{aligned} &\big(\dot{p}(t) - A^*\dot{p}(t + \Delta),\, \dot{q}(t - \Delta) - A^*\dot{q}(t)\big)\\ &\in \operatorname{co} D^*F(\bar{x}(t), \bar{x}(t - \Delta), \dot{\bar{x}}(t) - A\dot{\bar{x}}(t - \Delta))(-p(t) - q(t)) \quad \text{a.e. } t \in [a, b], \end{aligned} \tag{5.16}$$

Proof. Note that the Euler-Lagrange inclusions (5.5) is equivalently expressed in terms of the coderivative (3.4) with respect to (x,y), i.e., in the form

$$\begin{aligned}\big(\dot p(t) - A^*\dot p(t+\Delta),\, \dot q(t-\Delta) - A^*\dot q(t)\big) \in \operatorname{co} D^*F(\bar x(t), \bar x(t-\Delta),\\ \dot{\bar x}(t) - A\dot{\bar x}(t-\Delta))(-p(t)-q(t)),\ a.e.\ t\in[a,b].\end{aligned} \tag{5.17}$$

By Theorem 5.1 the optimal solution $\bar x(\cdot)$ satisfies conditions (5.1)–(5.5) and the relaxed counterpart of (5.17), which is the same as (5.16) in this case with F replaced by $\operatorname{co} F$. By Theorem 3.3 in Rockafellar [16] one has

$$\begin{aligned}&\operatorname{co}\big\{(u,v)\big|\ (u,w,p)\in N((x,y,v);\operatorname{gph}(\operatorname{co}F)\big\}\\ &= \operatorname{co}\big\{(u,w)\big|\ (-u,-w,v)\in\partial H_R(x,y,p,t)\big\},\end{aligned}$$

where H_R stands for the Hamiltonian of the relaxed system; i.e., with F replaced with the convexification $\operatorname{co} F$. It is easy to check that $H_R = H$. Thus the extended Euler-Lagrange inclusion for the relaxed system readily implies the necessary optimality conditions (5.14) and (5.15). When F is convex-valued, conditions (5.15) and (5.16) are *equivalent* due to the mentioned result of Rockafellar [16]. This completes the proof of the theorem. □

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