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Compactness of Solutions for Scalar Viscous Conservation Laws in Noncylindrical Domains

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Abstract. We utilize the kinetic formulation approach to study the compactness property for the family $\{u^\varepsilon\}_{\varepsilon>0}$, solutions of the initial-boundary value problem for the scalar viscous conservation law $u^\varepsilon_t + \mathrm{div}_x f(u^\varepsilon) = \varepsilon \Delta_x u^\varepsilon$ in a noncylindrical domain. Considering f in C^3 and satisfying the non-degeneracy condition, we prove that u^ε is compact in L^1_{loc}.

1. Introduction

Usually the partial differential equations which we observe have been guided by the perspective of Continuum Mechanics, almost obtained by systems of integral equations combining balance laws with constitutive relations. An alternative approach is the kinetic formulation, motivated by the classical kinetic theory of gases. In this approach, the state of a gas is described by the density function $\mathsf{f}(t,x,v)$ where t is the time, x the position and v the molecular velocity. The evolution of $\mathsf{f}(t,x,v)$ is governed by the Boltzmann equation, which monitors the changes in the distributional molecular velocities due to collisions and transport. The connections between the kinetic theory and the continuum approach are established by identifying intensive quantities like velocity, density, Cauchy stress tensor, temperature, etc., with appropriate moments of the density function $\mathsf{f}(t,x,v)$ and showing that these fields satisfy the balance laws of continuum physics.

In this context of kinetic theory, however established in an artificial way in which $\mathsf{f}(t,x,v)$ is allowed to take also negative values, we prove that the family $\{u^\varepsilon\}_{\varepsilon>0}$ solutions to the problem (1.1)-(1.3), i.e. an initial-boundary value problem for scalar viscous conservation laws, is compact in $L^1_{\mathrm{loc}}(Q_T)$. A usual procedure to obtain this result is to derive uniform estimates (with respect to the parameter $\varepsilon > 0$) on

$$\|\partial_t u^\varepsilon\|_{L^1(Q_T)} \text{ and } \|\nabla_x u^\varepsilon\|_{L^1(Q_T)}.$$

However, it seems impossible to derive such estimates for noncylindrical domains, even in the one-dimensional case, see Neves [9].

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We cite principally the Lions, Perthame & Tadmor [8] paper, who introduced the kinetic formulation and study the Cauchy problem for scalar conservation laws. Here, for the main result we follow Chen & Frid [1]. The reader could also see a complete reference to the kinetic theory in Perthame [10].

Let $f \in C^3(\mathbb{R};\mathbb{R}^n)$ be a given map, Q_T an open noncylindrical smooth domain of $\mathbb{R}^{n+1}$ whose points are denoted by $(t,x) \in \mathbb{R}\times\mathbb{R}^n$. We will denote Γ_T the lateral boundary of Q_T and by Ω the set $\overline{Q}_T \cap \{t=0\} \neq \emptyset$. Therefore for some $T>0$, we have

$$Q_T = \bigcup_{0<t<T} \Omega_t \times \{t\} \qquad , \qquad \Gamma_T = \bigcup_{0<t<T} \partial\Omega_t \times \{t\}.$$

For each $\varepsilon > 0$, we consider the initial-boundary value problem

$$u_t^\varepsilon + \mathrm{div}_x f(u^\varepsilon) = \varepsilon \Delta_x u^\varepsilon \qquad \text{in } Q_T, \tag{1.1}$$

$$u^\varepsilon = u_0 \qquad \text{in } \Omega, \tag{1.2}$$

$$u^\varepsilon = u_b \qquad \text{on } \Gamma_T. \tag{1.3}$$

We assume that the initial-boundary data satisfies

$$\begin{aligned} &u_0 \in H^1(\Omega) \cap L^\infty(\Omega;\mathcal{L}^n),\\ &u_b \in L^2(0,T;H^{3/2}(\partial\Omega_t)) \cap H^{3/4}(0,T;L^2(\partial\Omega_t)) \cap L^\infty(\Gamma_T;\mathcal{H}^n),\end{aligned}$$

where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure and $\mathcal{L}^n$ the n-dimensional Lebesgue measure. The initial-boundary value problem (1.1)-(1.3) has a unique solution

$$\begin{aligned} &u^\varepsilon \in L^2(0,T;H^2(\Omega_t)) \cap C(0,T;H^1(\Omega_t)) \cap L^\infty(Q_T),\\ &u_t^\varepsilon \in L^2(Q_T).\end{aligned}$$

This solution satisfies

$$\sup_{\varepsilon>0} |u^\varepsilon(t,x)| < \infty \quad \text{and} \quad \sup_{\varepsilon>0} \iint_D |\sqrt{\varepsilon}\nabla_x u^\varepsilon(t,x)|^2 \, dx\, dt < \infty,$$

where $D \subset Q_T$ is any compact set, see Neves [9].

The solution u^ε of (1.1) may exhibit boundary layers near Γ_T for $\varepsilon > 0$ small and appropriate boundary conditions. For instance, this happens when we try to solve the initial-boundary value problem to the scalar conservation law $u_t + \mathrm{div}_x f(u) = 0$, letting $\varepsilon \to 0^+$, see Serre [12] Vol. 2.

2. Equivalence with the Kinetic Formulation

We begin defining the density function f. Let $\chi : \mathbb{R}\times\mathbb{R} \to \{1,-1,0\}$,

$$\chi(\lambda,v) \equiv \chi_\lambda(v) := \begin{cases} +1, & \text{if } 0 \le v < \lambda \\ -1, & \text{if } \lambda \le v < 0 \\ \ \ 0, & \text{otherwise.} \end{cases}$$

Moreover, we have $\chi_\lambda(v) = [\,\mathrm{sgn}(\lambda - v) + \mathrm{sgn}(v)]/2$. Let $u(t,x)$ be a function defined in Q_T, and for $v \in \mathbb{R}$, we define

$$\mathsf{f}(t,x,v) := \chi_{u(t,x)}(v). \tag{2.1}$$

We observe that

$$u(t,x) = \int_{\mathbb{R}} \mathsf{f}(t,x,v)\,dv. \tag{2.2}$$

Now, we consider the following linear partial differential equation

$$\partial_t \mathsf{f} + a(v)\cdot\nabla_x \mathsf{f} - \Delta_x \mathsf{f} = \partial_v m \qquad \text{in } Q_T \times \mathbb{R} \tag{2.3}$$

for some nonnegative Radon measure $m(t,x,v)$, where $a(v) = f'(v)$.

The equation (2.3) is derived from the kinetic formulation as introduced in Lions, Perthame & Tadmor [8]. In this section, for simplicity we drop the parameter $\varepsilon > 0$. The next lemma shows that (1.1) and (2.3) are equivalent.

Lemma 2.1. *Let $u(t,x)$ be a function defined in $Q_T \subset \mathbb{R}^{n+1}$, and $\mathsf{f}(t,x,v)$ be defined by* (2.1)*. Then, u satisfies* (1.1) *in $\mathcal{D}'(Q_T)$ if and only if f satisfies* (2.3) *in $\mathcal{D}'(Q_T \times \mathbb{R})$.*

Proof. 1. First, with $u(t,x)$ a solution for (1.1), we show that f satisfies (2.3). For each $v \in \mathbb{R}$, let $F(\lambda,v) = (\eta(\lambda,v), q(\lambda,v))$ be a C^2 convex entropy pair. Then multiplying (1.1) by $\eta'(u(t,x),v)$, due that η is convex and the compatibility condition, i.e. $q' = \eta' f'$, we obtain

$$\partial_t \eta(u,v) + \mathrm{div}_x q(u,v) - \Delta_x \eta(u,v) \le 0 \qquad \text{in } \mathcal{D}'(Q_T). \tag{2.4}$$

For convenience, we denote

$$-L = \partial_t \eta(u,v) + \mathrm{div}_x q(u,v) - \Delta_x \eta(u,v)$$

and, for any non-negative function $\psi \in C_0^\infty(Q_T)$, we have

$$L(\psi) \ge 0. \tag{2.5}$$

Now as a consequence of the Schwartz lemma on negative distributions, see Schwartz [11], there exist a non-negative Radon measure m over $Q_T \times \mathbb{R}$ (moreover, uniformly bounded in case of m^ε), such that for any function $\phi \in C_0^\infty(Q_T)$

$$L(\phi) = \iint_{Q_T} \phi\, dm.$$

As usual, we do not distinguish between L and m, i.e.,

$$L(\phi) \equiv < L, \phi > = < m, \phi > = \iint_{Q_T} \phi\, dm.$$

Despite having taken $\eta \in C^2$, by approximation the above result is valid for the following entropy pair

$$\eta(\lambda, v) = \tfrac{1}{2}[|\lambda - v| - |v|],$$
$$q_i(\lambda, v) = \int_0^\lambda a_i(\xi)\partial_\xi \eta(\xi, v)\, d\xi \qquad (i = 1, \dots, n).$$

Then, we obtain

$$\begin{aligned}\partial_t[|u^\varepsilon - v| - |v|] + \operatorname{div}_x[(f(u^\varepsilon) - f(v))\operatorname{sgn}(u^\varepsilon - v) \\ +(f(0) - f(v))\operatorname{sgn}(v)] - \varepsilon\, \Delta_x[|u^\varepsilon - v| - |v|] = -2m^\varepsilon.\end{aligned} \tag{2.6}$$

Now, we differentiate (2.6) with respect to v in the sense of distributions

$$\begin{aligned}-\partial_t[\operatorname{sgn}(u - v) + \operatorname{sgn}(v)] - \operatorname{div}_x a(v)[\operatorname{sgn}(u - v) + \operatorname{sgn}(v)] \\ +\Delta_x[\operatorname{sgn}(u - v) + \operatorname{sgn}(v)] = -2\partial_v m \qquad \text{in } \mathcal{D}'(Q_T \times \mathbb{R}).\end{aligned}$$

Hence the result follows, that is, f satisfyies (2.3).
2. Now, if f satisfies (2.3) then we prove that $\eta(u)$ satisfies (2.4). Let η be a convex smooth entropy, at least a generalized one. Multiplying (2.3) by $\eta'(v)$, we get

$$\delta_{v=u}\eta'(v)\partial_t u + \delta_{v=u}\eta'(v)f'(v) \cdot \nabla_x u - \operatorname{div}_x[\delta_{v=u}\eta'(v)\nabla_x u] = \eta'(v)\partial_v m,$$

where δ_λ is the Dirac measure concentrated at λ. Moreover, taking η sublinear and integrating the above equation in $\mathbb{R}$ with respect to v, we obtain

$$\partial_t\eta(u) + \operatorname{div}_x q(u) - \Delta_x \eta(u) = -\int_{\mathbb{R}} \eta''(v)\, m\, dv \qquad \text{in } \mathcal{D}'(Q_T).$$

Since η is convex and m is a non-negative Radon measure, the result follows. □

3. Compactness of $\{u^\varepsilon\}$

In this section we are going to prove the L^1_{loc} compactness of the family $\{u^\varepsilon\}_{\varepsilon>0}$.

Theorem 3.1. *Let $f \in C^3(\mathbb{R}; \mathbb{R}^n)$ and satisfying the non-degeneracy condition, i.e.,*

$$\mathcal{L}^1(\{v \in \mathbb{R};\ \tau + a(v) \cdot k = 0\}) = 0,$$

for all $(\tau, k) \in \mathbb{R} \times \mathbb{R}^n$ with $\tau^2 + |k|^2 = 1$. Then the family $\{u^\varepsilon\}_{\varepsilon>0}$, solutions of (1.1)-(1.3), is compact in $L^1_{\text{loc}}(Q_T)$.

Proof. 1. For each $u^\varepsilon(t, x)$ and $v \in \mathbb{R}$, we set $\mathsf{f}^\varepsilon(t, x, v) = \chi_{u^\varepsilon(t,x)}(v)$. By (2.2) it is enough to prove that

$$\int_{\mathbb{R}} \mathsf{f}^\varepsilon(t, x, v)\, dv \qquad \text{is compact in } L^1_{\text{loc}}(Q_T). \tag{3.1}$$

Moreover, we observe that $\mathsf{f}^\varepsilon(t,x,v) \equiv 0$ when $|v| \geq R_0$, for any R_0 greater than the uniform boundness of u^ε in $L^\infty(Q_T)$. Hence the integral in (3.1) can be taken in a finite and fix interval $(-R_0, R_0)$.
2. For each $v \in (-R_0, R_0)$, let $F(\lambda, v) = (\eta(\lambda, v), q(\lambda, v))$ be an entropy pair, C^2 at least. From the proof of Lemma 2.1, we have

$$div_{t,x} F(u^\varepsilon(t,x), v) = \varepsilon \Delta_x \eta(u^\varepsilon(t,x), v) - m^\varepsilon. \tag{3.2}$$

Since u^ε is uniformly bounded, the left side of (3.2), viewed as a derivative of a bounded function, is uniformly bounded in $W^{-1,\infty}(Q_T \times \mathbb{R})$. Consequently,

$$\varepsilon \Delta_x \eta(u^\varepsilon, v) - m^\varepsilon \in \{\text{bounded of } W^{-1,\infty}(Q_T \times \mathbb{R})\}.$$

Moreover, m^ε has total variation uniformly bounded over any compact set contained in $Q_T \times \mathbb{R}$, that is, $m^\varepsilon \in \mathcal{M}_{\text{loc}}(Q_T \times \mathbb{R})$. Hence, applying Theorem 6 (Compactness for Measures) in Evans [3], we have

$$\{m^\varepsilon\} \quad \text{is pre-compact in } W^{-1,q}_{\text{loc}}(Q_T \times \mathbb{R}), \quad (1 \leq q < \frac{n+2}{n+1}).$$

Now, for any compact set $D \subset Q_T$ and $v \in (-R_0, R_0)$, $\varepsilon \Delta_x \eta(u^\varepsilon, v)$ is a continuous linear functional in $W^{-1,2}(Q_T \times \mathbb{R})$. Indeed, for any function $\phi \in W^{1,2}_0(Q_T \times \mathbb{R})$, we have

$$\begin{aligned} &| < \varepsilon \Delta_x \eta(u^\varepsilon, v), \phi > | = | \textstyle\int_{\mathbb{R}} \iint_{Q_T} \varepsilon \Delta_x \eta(u^\varepsilon, v) \phi \, dx \, dt \, dv| \\ &\leq C\sqrt{\varepsilon} (\textstyle\int_{-R_0}^{R_0} \iint_{Q_T} |\sqrt{\varepsilon} \nabla_x u^\varepsilon|^2)^{1/2} (\textstyle\int_{-R_0}^{R_0} \iint_{Q_T} |\nabla_x \phi|^2)^{1/2} \leq C\sqrt{\varepsilon} \|\nabla_x \phi\|_{L^2}, \end{aligned} \tag{3.3}$$

where C does not depend on ε. Taking the supremum in (3.3) with respect to the set $W := \{\phi \in W^{1,2}_0 : \|\phi\|_{W^{1,2}_0} \leq 1\}$, we obtain

$$\|\varepsilon \Delta_x \eta(u^\varepsilon, v)\|_{W^{-1,2}} \leq C\sqrt{\varepsilon} \sup_W |\phi|.$$

Therefore, passing to the limit as $\varepsilon \to 0^+$ in the above inequality, the family $\{\varepsilon \Delta_x \eta(u^\varepsilon, v)\}$ converges to zero in $W^{-1,2}_{\text{loc}}$, which implies that

$$\{\varepsilon \Delta_x \eta(u^\varepsilon, v)\} \quad \text{is pre-compact in } W^{-1,2}_{\text{loc}}(Q_T \times \mathbb{R}).$$

Now, $W^{-1,2}_{\text{loc}}(Q_T \times \mathbb{R})$ has continuous imbedding in $W^{-1,q}_{\text{loc}}(Q_T \times \mathbb{R})$ for any $1 \leq q < \frac{n+2}{n+1}$, it follows that

$$\begin{aligned} &\varepsilon \Delta_x \eta(u^\varepsilon, v) - m^\varepsilon \\ &\quad \in \{\text{bounded of } W^{-1,\infty}(Q_T \times \mathbb{R})\} \cap \{\text{pre-compact of } W^{-1,q}_{\text{loc}}(Q_T \times \mathbb{R})\}, \end{aligned}$$

and, applying Theorem 4.2 in Frid [7], we have

$$\varepsilon \Delta_x \eta(u^\varepsilon, v) - m^\varepsilon \in \{\text{pre-compact of } W^{-1,2}_{\text{loc}}(Q_T \times \mathbb{R})\}.$$

Then, we obtain

$$m^\varepsilon \in \{\text{pre-compact of } W^{-1,2}_{\text{loc}}(Q_T \times \mathbb{R})\}.$$

3. At this moment, we localize our problem in the following way. Given any compact set $D \subset Q_T$, we choose a smooth function ζ, such that ζ has compact support, $\zeta \equiv 1$ on D, $0 \leq \zeta \leq 1$. Then, multiplying (2.3) by ζ and denoting $\bar{\mathsf{f}}^\varepsilon(t,x,v) = \zeta(t,x)\, \mathsf{f}^\varepsilon(t,x,v)$, we have

$$\partial_t \bar{\mathsf{f}}^\varepsilon + a(v) \cdot \nabla_x \bar{\mathsf{f}}^\varepsilon - \varepsilon \Delta_x \bar{\mathsf{f}}^\varepsilon = \partial_v(\zeta\, m^\varepsilon) + \mathsf{f}^\varepsilon \zeta_t + \mathsf{f}^\varepsilon\, a(v) \cdot \nabla_x \zeta + 2\varepsilon \nabla_x \mathsf{f}^\varepsilon \cdot \nabla_x \zeta.$$

We would like to write the right side of the above equation, as a derivative with respect to v of a Radon measure in a compact set of $W^{-1,2}(Q_T \times \mathbb{R})$. Define

$$\overline{m}^\varepsilon := \zeta\, m^\varepsilon + \int^v \mathsf{f}^\varepsilon [\zeta_t + a(s) \cdot \nabla_x \zeta]\, ds + 2\varepsilon \int^v \nabla_x \mathsf{f}^\varepsilon \cdot \nabla_x \zeta\, ds. \tag{3.4}$$

For any function $\phi \in W^{1,2}_0(Q_T \times \mathbb{R})$, we have

$$\begin{aligned} |< \overline{m}^\varepsilon, \phi >| \leq |< m^\varepsilon, \zeta\phi >| &+ \textstyle\int_{\mathbb{R}} \iint_{Q_T} \int_{-R_0}^{R_0} |\mathsf{f}^\varepsilon|\, |\phi|\, |a(s)| |\nabla_{t,x} \zeta|\, ds\, dx\, dt\, dv \\ &+ 2\varepsilon \textstyle\int_{\mathbb{R}} \iint_{Q_T} \int_{-R_0}^{R_0} |\mathsf{f}^\varepsilon|\, |[\Delta_x \zeta \phi + \nabla_x \zeta \cdot \nabla_x \phi|\, ds\, dx\, dt\, dv \leq C\, \|\phi\|_{W^{1,2}_0}, \end{aligned}$$

where we have used that f^ε is uniformly bounded in $L^\infty(Q_T \times \mathbb{R})$, the compact support of ζ, and $m^\varepsilon \in W^{-1,2}_{\text{loc}}(Q_T \times \mathbb{R})$. So $\overline{m}^\varepsilon$ is a continuous linear functional in $W^{-1,2}(Q_T \times \mathbb{R})$. Moreover, passing to a subsequence, we observe that $< \overline{m}^\varepsilon, \phi >$ converges in $\mathbb{R}$. Then, we have $\overline{m}^\varepsilon \in \{$ pre-compact of $W^{-1,2}(Q_T \times \mathbb{R})\}$, and $\bar{\mathsf{f}}^\varepsilon$ satisfies

$$\partial_t \bar{\mathsf{f}}^\varepsilon + a(v) \cdot \nabla_x \bar{\mathsf{f}}^\varepsilon - \varepsilon \Delta_x \bar{\mathsf{f}}^\varepsilon = \partial_v \overline{m}^\varepsilon \qquad \text{em } \mathcal{D}'(Q_T \times \mathbb{R}). \tag{3.5}$$

For simplicity we maintain the notation f^ε, m^ε for $\bar{\mathsf{f}}^\varepsilon$ and $\overline{m}^\varepsilon$ respectively. Now, the support of m^ε is a fixed compact set contained in $Q_T \times \mathbb{R}$. Consequently, we can write, see Lions, Pertame and Tadmor [8],

$$\partial_v m^\varepsilon(t,x,v) = (I - \partial_v^2)(I - \Delta_{t,x})^{1/2} g^\varepsilon,$$

with g^ε belonging to a pre-compact set of $L^2(Q_T \times \mathbb{R})$. Then f^ε satisfies

$$\partial_t \mathsf{f}^\varepsilon + a(v) \cdot \nabla_x \mathsf{f}^\varepsilon - \varepsilon \Delta_x \mathsf{f}^\varepsilon = (I - \partial_v^2)(I - \Delta_{t,x})^{1/2} g^\varepsilon. \tag{3.6}$$

4. Let $\{\varepsilon_\ell\}_{\ell=1}^\infty$ be an arbitrary sequence which converges to zero as $\ell \to \infty$. Let g^{ε_ℓ} be a subsequence of g^ε, such that

$$g^{\varepsilon_\ell} \to \bar{g} \qquad \text{in } L^2(Q_T \times \mathbb{R}) \text{ when } \ell \to \infty$$

and, passing to a subsequence if necessary, we have

$$\mathsf{f}^{\varepsilon_\ell} \overset{*}{\rightharpoonup} \bar{\mathsf{f}} \qquad \text{in } L^\infty(Q_T \times \mathbb{R}) \text{ when } \ell \to \infty.$$

Clearly, $\bar{\mathsf{f}}$ and $\bar{g}$ satisfy (3.6). Moreover, we observe that $\mathsf{f}^{\varepsilon_\ell}$, $\bar{\mathsf{f}}$ have compact support in $Q_T \times (-R_0, R_0)$, so we have $\bar{\mathsf{f}}, \mathsf{f}^{\varepsilon_\ell} \in L^2(Q_T \times \mathbb{R})$. Now, we shall prove

$$\int_{\mathbb{R}} \mathsf{f}^{\varepsilon_\ell}(t,x,v)\, dv \to \int_{\mathbb{R}} \bar{\mathsf{f}}(t,x,v)\, dv \qquad \text{in } L^2(Q_T). \tag{3.7}$$

It is enough to show that, for any function $\psi \in C_0^\infty(-R_1, R_1)$ with $R_1 > R_0$

$$\int_{\mathbb{R}} \hat{\mathsf{f}}^{\varepsilon_\ell}(\tau, k, v)\psi(v)\, dv \to \int_{\mathbb{R}} \hat{\bar{\mathsf{f}}}(\tau, k, v)\psi(v)\, dv \qquad \text{in } L^2(\mathbb{R} \times \mathbb{R}^n), \tag{3.8}$$

where $\hat{\mathsf{f}}^{\varepsilon_\ell}(\tau, k, v)$ and $\hat{\bar{\mathsf{f}}}(\tau, k, v)$ are the Fourier Transform of $\mathsf{f}^{\varepsilon_\ell}(t, x, v)$ and $\bar{\mathsf{f}}(t, x, v)$ respectively. Indeed, by the Plancherel Identity, we obtain (3.7) taking $\psi \equiv 1$ over $(-R_0, R_0)$ in (3.8).
5. Let $\mathsf{f}(t, x, v), g(t, x, v) \in L^2(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$ satisfying (3.6), and $\bar{\delta} > 0$ to be choose a posteriori. We take $\zeta \in C_0^\infty(\mathbb{R})$, $\zeta \equiv 1$ in $(-1, 1)$, $\zeta \equiv 0$ in $\mathbb{R} \setminus (-2, 2)$ and following DiPerna & Lions [2], we write

$$\int_{\mathbb{R}} \hat{\mathsf{f}}(\tau, k, v)\, \psi(v)\, dv = I_1 + I_2; \tag{3.9}$$

$$I_1 = \int_{\mathbb{R}} \hat{\mathsf{f}}\psi(v)\zeta(\frac{\tau + a(v)\cdot k}{\bar{\delta}})\, dv, \quad I_2 = \int_{\mathbb{R}} \hat{\mathsf{f}}\psi(v)[1 - \zeta(\frac{\tau + a(v)\cdot k}{\bar{\delta}})]\, dv.$$

For $(\tau, k) \in \mathbb{R} \times \mathbb{R}^n$ with $\tau^2 + |k|^2 = 1$, we define the following distribution function

$$\mu(s) : [0, \infty) \to [0, \infty], \quad \mu(s) \equiv \mu^{\tau,k}(s) := \mathcal{L}^1(\{v \in (-R_1, R_1); |\tau + a(v)\cdot k| \le s\}).$$

Then, with $\tau^2 + |k|^2 > 0$ we have

$$\begin{aligned} |I_1| &= |\textstyle\int_{\mathbb{R}} \hat{\mathsf{f}}\, \psi(v)\, \zeta(\frac{\tau + a(v)\cdot k}{\bar{\delta}})\, dv| \\ &\le C\ (\textstyle\int_{|v|\le R_1} |\hat{\mathsf{f}}|^2)^{1/2}\ [\mu(\frac{\bar{\delta}}{(\tau^2+|k|^2)^{1/2}})]^{1/2}. \end{aligned} \tag{3.10}$$

Now, we estimate I_2. Taking the Fourier Transform in (3.6) with respect to (t, x), we get

$$\hat{\mathsf{f}}(\tau, k, v) = \frac{(1 + \tau^2 + |k|^2)^{1/2}}{\varepsilon|k|^2 + i(\tau + a(v)\cdot k)}(I - \partial_v^2)\hat{g}(\tau, k, v).$$

Then, we have

$$\begin{aligned} |I_2| \le \quad & C\,(1 + \tau^2 + |k|^2)^{1/2} \left(\textstyle\int_{|v|\le R_1} |\hat{g}|^2\, dv\right)^{1/2} (\textstyle\int_{|v|\le R_1} 1_{\{|\tau + a(v)\cdot k| \ge \bar{\delta}\}} \\ & \left\{ \tfrac{1 + |k|^2/\bar{\delta}^2 + |k|^4/\bar{\delta}^4}{|\tau + a(v)\cdot k|^2} + \tfrac{|k|^2 + |k|^4/\bar{\delta}^2}{|\tau + a(v)\cdot k|^4} + \tfrac{|k|^4}{|\tau + a(v)\cdot k|^6} \right\}\, dv)^{1/2}. \end{aligned} \tag{3.11}$$

Now, observing that

$$\int_{-R_1}^{R_1} \frac{1_{\{|\tau + a(v)\cdot k| \ge \bar{\delta}\}}}{|\tau + a(v)\cdot k|^p}\, dv = \frac{1}{(\tau^2 + |k|^2)^{p/2}} \int_{\frac{\bar{\delta}}{(\tau^2+|k|^2)^{1/2}}}^{\infty} \frac{1}{s^p}\, d\mu(s) \le C\, \bar{\delta}^{-p},$$

see Folland [6] (Distribution Functions), we obtain from (3.11)

$$|I_2| \le C\,(1 + \tau^2 + |k|^2)^{1/2} \left(\int_{|v|<R_1} |\hat{g}|^2\, dv\right)^{1/2} (\bar{\delta}^{-1} + |k|\bar{\delta}^{-2} + |k|^2\bar{\delta}^{-3}). \tag{3.12}$$

6. Finally we prove (3.8), that is,

$$\iint_{\mathbb{R} X \mathbb{R}^n} \Big| \int_{\mathbb{R}} (\hat{\bar{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell})\, \psi(v)\, dv \Big|^2 \, d\tau \, dk \underset{\ell\to\infty}{\longrightarrow} 0.$$

Since $\hat{\bar{\mathsf{f}}}, \hat{\mathsf{f}}^{\varepsilon_\ell} \in L^2(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R})$, given $\delta > 0$, there exist $\gamma > 0$, such that

$$\iint_{(\tau^2+|k|^2)^{1/2}<\gamma} \int_{|v|<R_1} |(\hat{\bar{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell})|^2 \, dv \, d\tau \, dk < \delta. \tag{3.13}$$

Fixed $\delta > 0$, we prove that

$$\iint_{(\tau^2+|k|^2)^{1/2}\geq\gamma} \Big| \int_{\mathbb{R}} (\hat{\bar{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell})\, \psi(v)\, dv \Big|^2 \, d\tau \, dk \underset{\ell\to\infty}{\longrightarrow} 0, \tag{3.14}$$

where we utilize (3.9), (3.10) and (3.12) with f and g replaced by $\bar{\mathsf{f}} - \mathsf{f}^{\varepsilon_\ell}$ and $\bar{g} - g^{\varepsilon_\ell}$ respectively. Indeed, for $(\tau^2 + |k|^2)^{1/2} \geq \gamma$, we choose $\bar{\delta} = \delta'(\tau^2 + |k|^2)^{1/2}$, with $\delta' > 0$ a small number. Then by (3.10) and (3.12), we have

$$|I_1| \leq C \, [\mu^{\tau,k}(\delta')]^{1/2} \Big(\int_{|v|<R_1} |\hat{\bar{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell}|^2 \, dv \Big)^{1/2},$$

$$|I_2| \leq C \, h(\tau,k) \, [(\delta')^{-1} + (\delta')^{-2} + (\delta')^{-3}] \, \Big(\int_{|v|<R_1} |\hat{\bar{g}} - \hat{g}^{\varepsilon_\ell}|^2 \, dv \Big)^{1/2},$$

where $h(\tau,k) \in L^\infty(\{\tau^2 + |k|^2 \geq \gamma\})$. Consequently,

$$\begin{aligned}
&\iint_{(\tau^2+|k|^2)^{1/2}\geq\gamma} \Big| \int_{\mathbb{R}} (\hat{\bar{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell})\, \psi(v)\, dv \Big|^2 \, d\tau \, dk \\
&\quad \leq C_1 \iint_{(\tau^2+|k|^2)^{1/2}\geq\gamma} [\mu^{\tau,k}(\delta')] \int_{|v|<R_1} |\hat{\bar{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell}|^2 \, dv \;\, d\tau \, dk \\
&\qquad + C_2 \sup_{(\tau^2+|k|^2)^{1/2}} |h(\tau,k)|^2 \iint_{(\tau^2+|k|^2)^{1/2}\geq\gamma} \int_{|v|<R_1} |\hat{\bar{g}} - \hat{g}^{\varepsilon_\ell}|^2 \, dv \, d\tau \, dk.
\end{aligned}$$

The Plancherel identity implies that

$$\hat{g}^{\varepsilon_\ell} \to \hat{\bar{g}} \qquad \text{in } L^2(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}),$$

so the second term in the right side of the above inequality goes to zero. For the former, we have the following. Let $\varpi : S^n \times [0,1] \to [0,\infty]$, $\varpi(\tau,k,s) := \mu^{\tau,k}(s)$. It is not difficult to show that ϖ is uniformly continuous in $S^n \times [0,1]$. Indeed, since $S^n \times [0,1]$ is a compact set, it is enough to prove that ϖ is continuous. Let $(\tau_\ell, k_\ell, s_\ell) \in S^n \times [0,1]$ be a convergent sequence, such that

$$(\tau_\ell, k_\ell, s_\ell) \underset{\ell\to\infty}{\longrightarrow} (\tau_\omega, k_\omega, s_\omega) \in S^n \times [0,1].$$

Then, applying the Dominated Convergence Theorem, we get

$$|\mu^{\tau_\ell,k_\ell}(s_\ell) - \mu^{\tau_\omega,k_\omega}(s_\omega)| \leq \int_{-R_1}^{R_1} |1_{\{|\tau_\ell + a(v)\cdot k_\ell| \leq s_\ell\}} - 1_{\{|\tau_\omega + a(v)\cdot k_\omega| \leq s_\omega\}}| \, dv \underset{\ell\to\infty}{\longrightarrow} 0,$$

which implies that ϖ is continuous in $S^n \times [0,1]$. Moreover, we have

$$\varpi(\tau,k,0) = \mathcal{L}^1(\{v \in (-R_1,R_1); |\tau + k\cdot a(v)| = 0\}) = 0,$$

for all $(\tau,k) \in S^n$. Hence we obtain that

$$\iint_{(\tau^2+|k|^2)^{1/2}\geq\gamma} [\mu^{\tau,k}(\delta')] \int_{|v|<R_1} |\hat{\hat{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell}|^2 \, dv \;\, d\tau \, dk$$
$$\leq \sup_{(\tau,k)\in S^n} \mu^{\tau,k}(\delta') \iint_{(\tau^2+|k|^2)^{1/2}\geq\gamma} \int_{|v|<R_1} |\hat{\hat{\mathsf{f}}} - \hat{\mathsf{f}}^{\varepsilon_\ell}|^2 \, dv \, d\tau \, dk$$

and, since δ' can be take arbitrary small, the final result follows. □

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Resumo. Utilizamos a teoria cinética para estudar a compacidade da família $\{u^\varepsilon\}_{\varepsilon>0}$, soluções do problema de valor inicial e de contorno para leis de conservação escalares viscosas $u^\varepsilon_t + \mathrm{div}_x f(u^\varepsilon) = \varepsilon\Delta_x u^\varepsilon$ em domínios não cilíndricos. Considerando f em C^3 e satisfazendo à condição de ser não degenerada, provamos que u^ε é compacta em L^1_{loc}.

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