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The Thermostat Problem

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Abstract. A paradigm model for an air-conditioning system is studied: heat flux to and from one end of a bar is a (nonlinear) function of the temperature at the other end. The behaviour of this model is studied through semi-discretisation in the spatial variable and local linearisation. This procedure produces an autonomous system of ordinary differential equations whose local stability may be studied through its spectrum of eigenvalues and related back to the original problem through the Hopf bifurcation theorem. It will be shown that the proportionality constant γ is a bifurcation parameter which gives rise to three qualitatively different solutions: one stable, where the temperature tends exponentially to zero; one stable that is bounded by an envelope which tends exponentially to zero; and an unstable solution that oscillates with ever increasing amplitude. An almost local problem is also studied with similar results: the three qualitative solutions arise as before with the bifurcation parameter decreasing as the problem becomes closer to the local problem.

Integral equation characterisations of the nonlinear problem are developed and existence and uniqueness are demonstrated. For the linear problem the general analytic solution is provided and its numerical evaluation is discussed.

1. Introduction

A rod of unit length lying on the x-axis has its ends at $x = 0$ and $x = 1$. Its sides are perfectly insulated so that no heat can enter or escape through them. At time $t = 0$, the temperature of the rod is given by $\Phi(x), 0 \le x \le 1$. For $t > 0$, the left end ($x = 0$) of the rod is insulated, and heat is added or extracted at the right end ($x = 1$) as a function of the temperature at the left end. The initial-boundary value problem for the temperature $u(x,t)$ in the rod is

$$\begin{aligned} u_t &= u_{xx}, && 0 < x < 1,\, t > 0, && (1.1a)\\ u(x,0) &= \Phi(x), && 0 \le x \le 1, && (1.1b)\\ u_x(0,t) &= 0, && t > 0, && (1.1c)\\ u_x(1,t) &= H(u(0,t)), && t > 0, && (1.1d) \end{aligned}$$

where H is a nonlinear function such that $H(0) = 0$. Another boundary condition of interest is

$$u_x(1,t) = H(u(x^*,t)), \qquad t > 0, \tag{1.2}$$

where $x^* \in \langle 0, 1)$. In the case when flux leakage is permitted at the left end the boundary condition

$$u_x(0,t) = \beta(u(0,t)+1), \qquad t > 0 \tag{1.3}$$

is considered instead of (1.1c). The special case of the linear boundary condition

$$u_x(1,t) = -\gamma u(0,t), \qquad t > 0, \tag{1.4}$$

where $\gamma > 0$, will also be discussed. When heat loss is permitted the non-homogeneous equation

$$u_t = u_{xx} - \alpha u$$

will be considered instead of (1.1a).

Problem (1.1) may be interpreted as a model for a thermostat with the sensor and controller positioned at opposite ends of an interval. Problem (1.1) with boundary condition (1.2) is the more general case of the nonlinear problem; the controller is positioned at some arbitrary point other than the heat input. Both are nonlocal problems.

Our motivation for studying this problem comes from a paper written by Guidotti and Merino [7], where a similar initial-boundary value problem was associated with an abstract Cauchy problem using the general results presented by Amann [1]: in this work the principle of linearized stability (see Drangeid [4]) was applied and the spectrum of the associated linear operator was investigated through the Hopf bifurcation theorem (see e.g. Guckenheimer & Holmes [6]). In a follow-up paper Guidotti and Merino [8] studied the invariance properties of the model for a thermostat. The dynamical behaviour of the solution of similar parameter-dependent reaction-diffusion equations has also been studied by Simonett [15, 16] within a more general setting. Problems closely related to (1.1) have been studied by Friedman and Jiang [5] and by Brokate and Friedman [2].

To resolve questions of existence and uniqueness of a solution of an initial-boundary value problem, a common strategy is to seek an equivalent integral (or an integro-differential) equation reformulation. This equivalence may then be employed to analyse the original problem. Many initial-boundary value problems involving the heat equation can be transformed into a Volterra integral equation (see e.g. Cannon [3]) and this strategy for studying nonlocal nonlinear problems has been employed by several authors (see e.g. Jumarhon [9]; Jumarhon & McKee [10]; Lin [11, 12]).

In §2 we recast problem (1.1) as a Volterra integral equation. The existence and uniqueness of the original problem is then demonstrated through the existence and uniqueness of the solution of the associated Volterra equation. The same Volterra integral equation may be obtained from the application of Laplace transforms to (1.1) and through this technique we present, in §3, another two integral relationships.

Semi-discretisation of the spatial variable allows us to rewrite problem (1.1) as an autonomous system of ordinary differential equations and in §4 we show that a solution $\underline{u} = (u_0, \ldots, u_N)$ of this autonomous system tends to the solution of (1.1) as N tends to infinity. We can thus employ a well-known approach from ordinary differential equations: investigate the behaviour of the solution of the nonlinear system from the eigenvalues of associated linearisation. This results in a linear parameter-dependent autonomous system of ordinary differential equations. The parameter, $-\gamma$, which may be identified as $\partial H(0)/\partial u$, plays the role of a bifurcation parameter and determines the qualitative nature of the solution of the nonlinear problem through the eigenvalues (as $N \to \infty$) of the associated linear system of ordinary differential equations.

Laplace transforms techniques allows us to write down the analytic solution of the linearized problem (i.e. (1.1) with (1.4)) directly in terms of the sum of residues (see §5). A study is then made of the pole positions. Three qualitatively different solutions are found dependent upon the value of γ. In particular, if $\gamma > 17.798542$, it is seen that the temperature of the rod will be unstable and will oscillate with ever increasing amplitude. It is also shown that the eigenvalues of the linearized autonomous system of the ordinary differential equations tend to the poles as $N \to \infty$.

In §6 Laplace transform techniques are applied to the almost local linear problem. The same qualitative features are maintained although the value of γ required to produce an unstable temperature profile increases as x^* gets closer to $x = 1$. In §7 the same approach is used to investigate the linear problem with flux leakage, characterised by the heat transfer coefficient β. As β increases instability is still displayed albeit with larger values of γ. Finally §8 deals with the issue of heat loss from the bar.

2. A Volterra Integral Equation Reformulation

To deduce the existence and uniqueness of the solution of the initial-boundary value problem (1.1) we derive an equivalent integral equation to which existence and uniqueness theory can be applied. This integral equation approach is similar to that used by Cannon [3]. Indeed by modifying Theorem 6.4.1 of Cannon [3] and applying it directly to (1.1) we can deduce:

Theorem 2.1. *For piecewise-continuous Φ and continuous H, the initial-boundary value problem (1.1) has a unique solution*

$$u(x,t) = \int_0^1 (\theta(x-\xi,t) + \theta(x+\xi,t))\,\Phi(\xi)\,d\xi + 2\int_0^t \theta(x-1,t-\tau)H(u(0,\tau))\,d\tau, \tag{2.1}$$

where $\theta(x,t)$ is defined by

$$\theta(x,t) = \sum_{n=-\infty}^{\infty} K(x+2n,t), \qquad t > 0,$$

if and only if $u(0,t)$ is the unique piecewise-continuous solution of

$$u(0,t) = 2\left(\int_0^1 \theta(\xi,t)\Phi(\xi)\,d\xi + \int_0^t \theta(1,t-\tau)H(u(0,\tau))\,d\tau\right). \tag{2.2}$$

Once the solution of (2.2) is known, $u(x,t)$ can be determined using (2.1). The following theorem is a direct consequence of checking that the conditions in Theorem 8.2.1 of Cannon [3] are satisfied.

Theorem 2.2. *The Volterra integral equation (2.2) has a unique continuous solution on* $(0,\infty)$.

3. The Integral Equation Reformulation

Using the Laplace transform with respect to time

$$\bar{u}(x,p) = \int_0^\infty u(x,t)e^{-pt}dt, \tag{3.1}$$

and the convolution theorem

$$\mathcal{L}^{-1}\Big(\mathcal{L}(f(t))\ \mathcal{L}(g(t))\Big) = (f*g)(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

allows the original problem to be characterised by three different, but related, integral relationships. They are: an integral relationship between $u(0,t)$ and $u(0,\tau)$ given by (2.2); an integral relationship between $u(1,t)$ and $u(0,\tau)$

$$u(1,t) = 2\ \left(\int_0^1 \theta(\xi-1,t)\Phi(\xi)\ d\xi + \int_0^t \theta(0,t-\tau)H(u(0,\tau))\ d\tau\right);$$

and an integral relationship between $u(0,t)$ and $u(1,\tau)$

$$u(0,t) = \int_0^t \frac{u(1,\tau)}{\sqrt{\pi(t-\tau)^3}}\sum_{n=0}^\infty (-1)^n(2n+1)\exp\left(-\frac{(2n+1)^2}{4(t-\tau)}\right)\ d\tau$$

$$+\int_0^1 \frac{\Phi(\xi)}{\sqrt{\pi t}}\sum_{n=0}^\infty (-1)^n\left[\exp\left(-\frac{(2n+\xi)^2}{4t}\right) - \exp\left(-\frac{(2(n+1)-\xi)^2}{4t}\right)\right]\ d\xi,$$

which may be written as

$$u(0,t) = \int_0^t u(1,\tau)k(t-\tau)\ d\tau + G(t),$$

with obvious notation.

4. Semi-discretisation to an Autonomous System

We use a semi-discretisation technique (see e.g. Zafarullah [18]; Smith [17]; Schiesser [13]) to reduce the initial-boundary value problem (1.1) to an autonomous system of ordinary differential equations

$$\underline{u}_t = \mathcal{A}(\underline{u}), \tag{4.1}$$

where $\mathcal{A}$ is a nonlinear operator and $\underline{u} = (u_0, \dots, u_N)$, where u_i is an approximation of $u(x_i, t)$ for $x_i = i\triangle x, i = 0, 1, \dots, N$, where $\triangle x = 1/N$. Thus for the initial-boundary value problem (1.1) the system (4.1) takes the form

$$\begin{aligned}
\dot{u}_0 &= \frac{1}{(\triangle x)^2}(-2u_0 + 2u_1), \\
\dot{u}_1 &= \frac{1}{(\triangle x)^2}(u_0 - 2u_1 + u_2), \\
&\vdots \\
\dot{u}_{N-1} &= \frac{1}{(\triangle x)^2}(u_{N-2} - 2u_{N-1} + u_N), \\
\dot{u}_N &= \frac{1}{(\triangle x)^2}(2\triangle x H(u_0) + 2u_{N-1} - 2u_N),
\end{aligned} \tag{4.2}$$

where $\dot{u}_i$ stands for du_i/dt. The equivalence of a solution of (1.1) with a solution of (4.2) as $N \to \infty$ has already been demonstrated (see e.g. Zafarullah [18]). It allows us to treat the original problem as the limit of the above autonomous system.

To analyze the nonlinear system (4.2) we have to determine the critical points $\underline{u}_c$ of $\mathcal{A}(\underline{u}) = 0$, and infer their stability from the behaviour of the linearized system

$$\dot{\underline{u}} = A\underline{u}$$

at these points. Here A is the Jacobian and takes the form

$$A = \frac{1}{(\triangle x)^2}\begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 2\triangle x H' & & & 2 & -2 \end{pmatrix}_{(N+1)\times(N+1)},$$

where $H' = \partial H(\underline{u}_c)/\partial u_0$. However, it is clear that the only critical point occurs when $\underline{u}_c = 0$. Furthermore

$$H(u(0,t)) = H(0) + \frac{\partial H(0)}{\partial u}u(0,t) + O(u(0,t)^2) = \frac{\partial H(0)}{\partial u}u(0,t) + O(u(0,t)^2),$$

and hence we may identify $\partial H(0)/\partial u$ as $-\gamma$. Thus, in order to understand the stability of problem (1.1), we need to study the spectrum of the following parameter-dependent matrix

$$A(\gamma) = \frac{1}{(\triangle x)^2}\begin{pmatrix} -2 & 2 & & & & \\ 1 & & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ -2\triangle x\gamma & & & & 2 & -2 \end{pmatrix}_{(N+1)\times(N+1).} \tag{4.3}$$

Generally, if the matrix (4.3) has all of its eigenvalues in the open left-half plane, $Re\lambda < 0$, it follows that $\underline{u}_c = 0$ is asymptotically stable. If there exists an eigenvalue λ such that $Re\lambda > 0$ then the critical point $\underline{u}_c = 0$ is unstable. If a pair of conjugate purely imaginary eigenvalues occurs and all the remaining eigenvalues satisfy $Re\lambda < 0$ then the Hopf bifurcation theorem can be employed to deduce the qualitative behaviour of the associated nonlinear problem (1.1).

It is straightforward to show that the $N+1$ eigenvalues and eigenvectors are

$$\lambda_k = 2N^2(\cos\varphi_k - 1), \qquad v_{jk} = v_0 \cos j\varphi_k \qquad j = 0, 1, \ldots, N,$$

where φ_k are the $N+1$ roots of

$$N \sin\varphi \sin N\varphi = \gamma.$$

5. Application of the Laplace Transform Method

Up to this point we have been dealing mainly with the nonlinear problem (1.1). In the previous section we have argued that we can replace $H(u(0,t))$ by the parameter $-\gamma$ and thereby study a linear parameter-dependent problem. In this section the model with the linear nonlocal boundary condition (1.4) (in place of (1.1d)) will be considered. Laplace transforms combined with Cauchy's theorem allows us to write down the analytic solution as a sum of residues. The poles associated with the residues will be determined and it can be shown that

only a finite, indeed quite a small number need to be considered in order to compute a reasonably accurate numerical solution.

On applying the Laplace transform (3.1) to the partial differential equation (1.1a) and the boundary conditions (1.1c) and (1.4) we obtain

$$\begin{aligned} \bar{u}_{xx} - p\bar{u} &= -\Phi(x), & (5.1a)\\ \bar{u}_x(0,p) &= 0, & (5.1b)\\ \bar{u}_x(1,p) &= -\gamma\bar{u}(0,p). & (5.1c) \end{aligned}$$

On solving we obtain

$$\bar{u}(x,p) = \frac{\cosh(\sqrt{p}x)}{\sqrt{p}\sinh\sqrt{p} + \gamma}\int_0^1 \Phi(\xi)\cosh(\sqrt{p}(1-\xi))\,d\xi - \frac{1}{\sqrt{p}}\int_0^x \Phi(\xi)\sinh(\sqrt{p}(x-\xi))\,d\xi.$$

This is the solution of the problem (5.1) in Laplace transform space. We determine $u(x,t)$ from $\bar{u}(x,p)$ using the complex inversion formula

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{zt}\bar{u}(x,z)\,dz \\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{zt}\left(\frac{\cosh(\sqrt{z}x)}{\sqrt{z}\sinh\sqrt{z}+\gamma}\int_0^1 \Phi(\xi)\cosh(\sqrt{z}(1-\xi))\,d\xi \right. \\ &\quad \left. -\frac{1}{\sqrt{z}}\int_0^x \Phi(\xi)\sinh(\sqrt{z}(x-\xi))\,d\xi\right)\,dz. \end{aligned} \tag{5.2}$$

The point $z=0$ is a removable singularity in the second integral of (5.2). Consequently the only zeros of the denominator of the first integral of (5.2) are given by

$$\sqrt{z}\sinh\sqrt{z}+\gamma=0. \tag{5.3}$$

5.1. Determination of the poles

To determine the zeros z_n of (5.3) let $\sqrt{z}=a+ib$ ($z=a^2-b^2\pm 2iab$). Then (5.3) can be written in the form

$$(a+ib)\sinh(a+ib)+\gamma=0$$

where the real values a and b are solutions of the equations

$$a\sinh a\cos b - b\cosh a\sin b = -\gamma, \tag{5.4}$$

$$b\sinh a\cos b + a\cosh a\sin b = 0. \tag{5.5}$$

Since the left-handrm sides of both (5.4) and (5.5) are even functions we need only consider the case $a\geq 0$ and $b\geq 0$. Note that for $\gamma=0$ the zeros of (5.3) are $z_n=-n^2\pi^2, n=0,1,2,\ldots$. We shall now study the case $\gamma>0$.

If $b=0$ then (5.3) has positive real roots $z=a^2$, where a are the roots of

$$a\sinh a=-\gamma. \tag{5.6}$$

The left-hand side of (5.6) is positive while the right-hand side is negative. Thus, there is no positive real pole for any $\gamma>0$.

If $a=0$ then (5.3) has the negative real roots $z=-b^2$, where b are the roots of

$$b\sin b=\gamma. \tag{5.7}$$

Figure 1 shows that there is an infinite number of negative real poles of order one for every value of γ. Moreover, for some values of γ we can observe the existence of negative real poles of order two. In this case γ and the corresponding b satisfy not only (5.3) but also

$$\frac{d}{dz}(\sqrt{z}\sinh\sqrt{z}+\gamma)=0$$

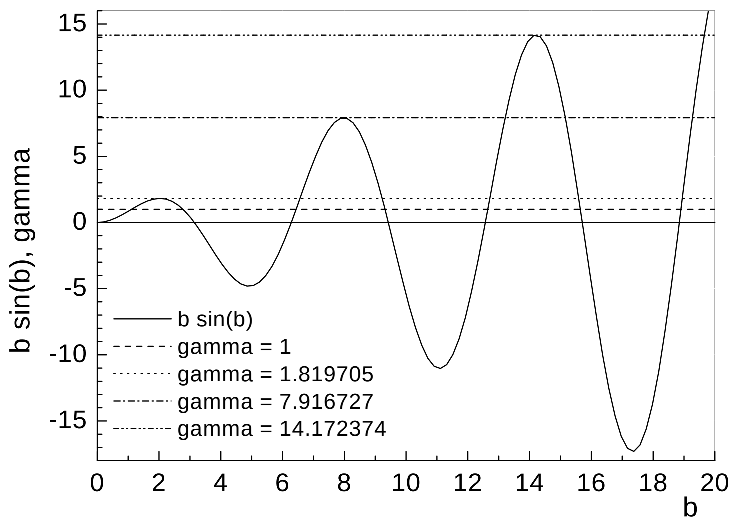


Figure 1: Determination of negative real poles $z = -b^2$ of order one and two.

For these values of γ the functions $b \sin b$ and γ on the left-hand and right-hand sides of the equation (5.7) have the common tangent, i.e.

$$\sin b + b \cos b = 0. \tag{5.8}$$

Once we have found the roots of (5.8), we obtain from (5.7) those values of γ which permit double roots to occur. The smallest such value is $\gamma = 1.819705$ and, in addition to this, Figure 1 depicts another two values of γ when double roots occur.

Further substantial analysis shows that as γ increases the roots come together in pairs and coalesce sequentially starting with the pair closest to the imaginary axis. These double roots then split and travel into the complex plane along an arc before eventually crosing the imaginary axis. It is demonstrated that this will occur when $\gamma = 17.798542$. In fact this can be deduced from Figure 1: as γ increases we observe a sequence of double roots being created and then disappearing.

The critical values of the parameter γ ($\gamma = 1.819705$ and $\gamma = 17.798542$) coincide with the ones obtained by Guidotti & Merino in [7]. In their paper the equivalent values of γ are 0.5792 and 5.6655 since their problem is defined over $(0, \pi)$ rather then $(0, 1)$.

Finally an equivalence may be established between the poles and the eigenvalues of the semi-discretised problem of §4 as $N \to \infty$. This allows us to determine the stability of the original thermostat problem from an examination of the poles.

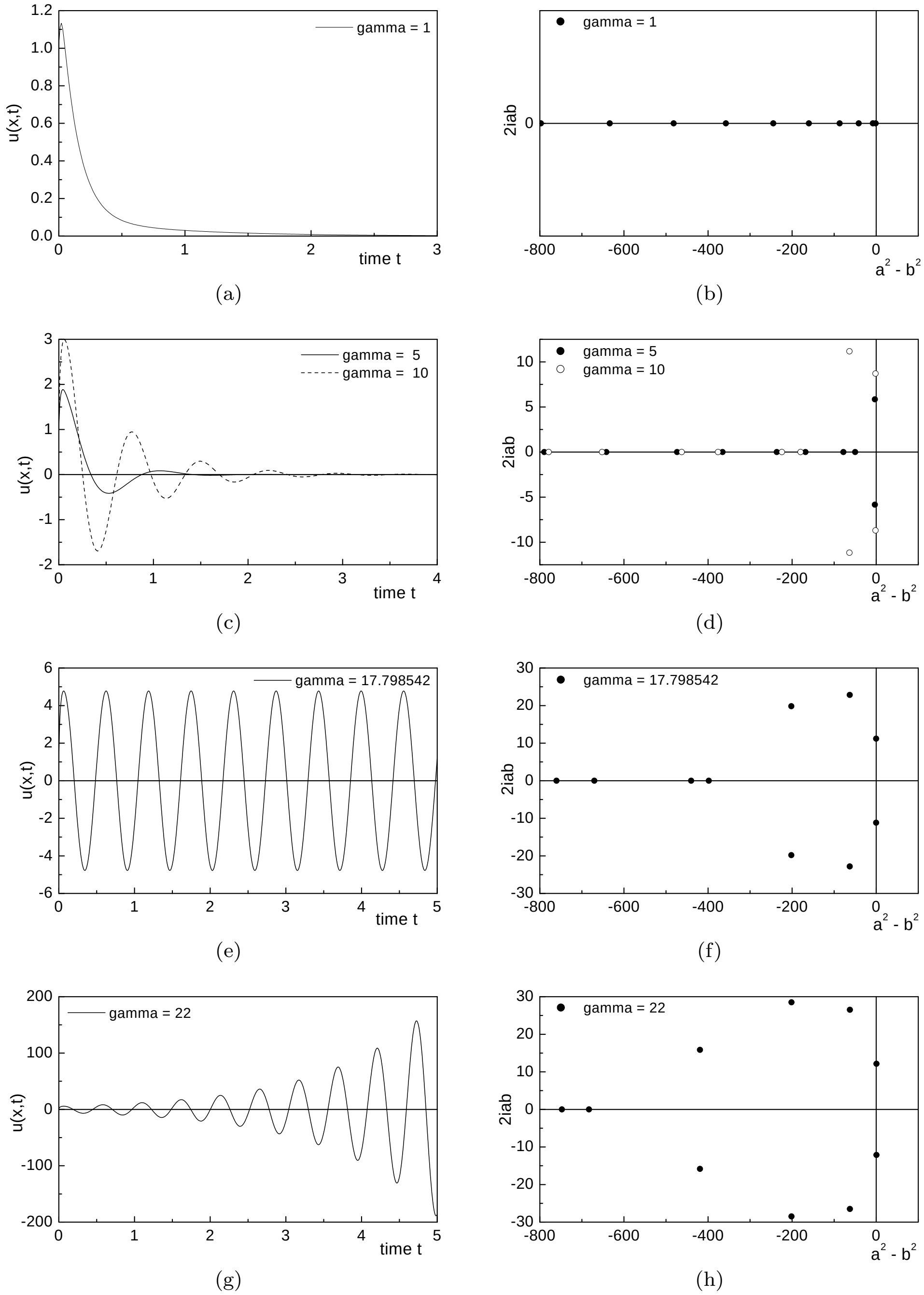
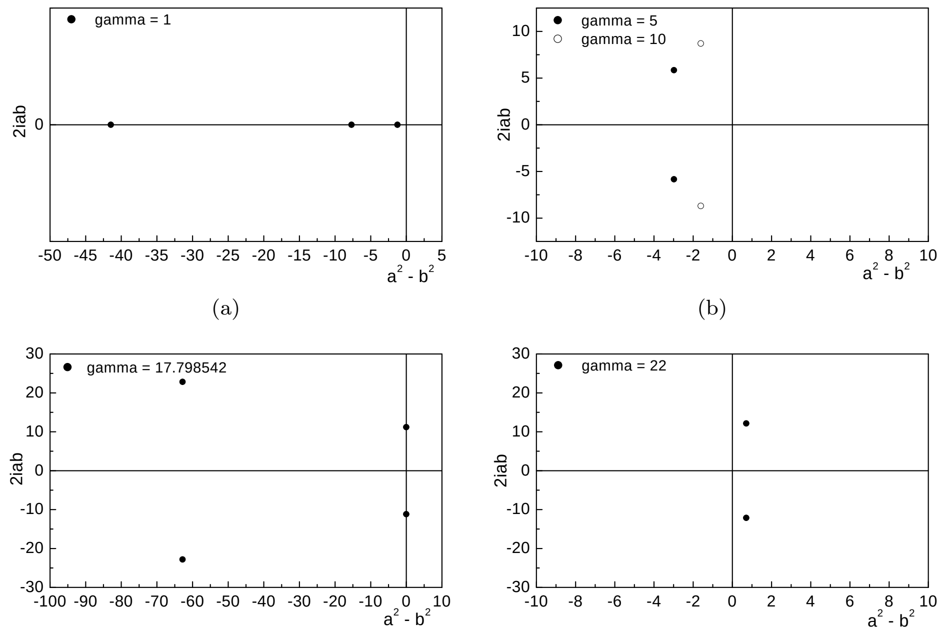


Figure 2: Numerical plots of $u(x,t)$ computed from the analytic solution for various values of γ and the position of their corresponding poles in the complex plane.



(a) (b) (c) (d)

Figure 3: Poles near the imaginary axis. A magnification of Figure 3.

5.2. Analytic solution

We can write the solution (5.2) as the sum of residues

$$u(x,t) = \sum_{\forall n} \frac{2e^{z_n t}\sqrt{z_n}\cosh(\sqrt{z_n}x)}{\sinh\sqrt{z_n} + \sqrt{z_n}\cosh\sqrt{z_n}} \int_0^1 \Phi(\xi)\cosh(\sqrt{z_n}(1-\xi))\, d\xi. \tag{5.9}$$

If γ takes the value for which double poles occur, we add to $u(x,t)$ in (5.9) a further sum of residues corresponding to these poles. This solution and position of its asociated poles is graphically illustrated by Figures 2 and 3.

6. Almost Local Problem

In this section we turn our attention to the problem (1.1a-c) with the boundary condition (1.2). The behaviour of the solution of the almost local initial-boundary value problem with $x^* = 1-\epsilon, 0 < \epsilon \ll 1$ can be investigated using the technique of the preceding sections. Note that a similar linearisation may be carried out and we can write down the solution of the associated linear problem in Laplace transform space

$$\begin{aligned}\bar{u}(x,p) \quad = \quad & \frac{\gamma\cosh(\sqrt{p}x)}{\sqrt{p}(\sqrt{p}\sinh\sqrt{p}+\gamma\cosh(\sqrt{p}x^*))}\int_0^{x^*}\Phi(\xi)\sinh(\sqrt{p}(x^*-\xi))\,d\xi \\ & +\frac{\cosh(\sqrt{p}x)}{\sqrt{p}\sinh\sqrt{p}+\gamma\cosh(\sqrt{p}x^*)}\int_0^1\Phi(\xi)\cosh(\sqrt{p}(1-\xi))\,d\xi \\ & -\frac{1}{\sqrt{p}}\int_0^x\Phi(\xi)\sinh(\sqrt{p}(x-\xi))\,d\xi;\end{aligned}$$

thus $u(x,t)$ is given by

$$\begin{aligned}u(x,t) \; = & \textstyle\sum_{\forall n}\frac{2e^{z_nt}\gamma\cosh(\sqrt{z_n}x)}{\sinh\sqrt{z_n}+\sqrt{z_n}\cosh\sqrt{z_n}+x^*\gamma\sinh(\sqrt{z_n}x^*)}\int_0^{x^*}\Phi(\xi)\sinh(\sqrt{z_n}(x^*-\xi))\,d\xi \\ & +\textstyle\sum_{\forall n}\frac{2e^{z_nt}\sqrt{z_n}\cosh(\sqrt{z_n}x)}{\sinh\sqrt{z_n}+\sqrt{z_n}\cosh\sqrt{z_n}+x^*\gamma\sinh(\sqrt{z_n}x^*)}\int_0^1\Phi(\xi)\cosh(\sqrt{z_n}(1-\xi))\,d\xi\end{aligned}$$

plus any residues corresponding to the poles of order two.

Substantial analysis shows that the same pattern emerges for the almost local case as existed for the nonlocal case. Here there exist ranges of γ, $(0,\gamma_1),(\gamma_1,\gamma_2)$, (γ_2,∞) for which the solution is respectively, exponentially stable, oscillatory but bounded, and unstable. As might have been anticipated the closer x^* is to 1 the more stable the system will be i.e. the critical values γ_1 and γ_2 increase as $x^*\to 1$.

7. A Model with the Flux Leakage from Left End

In the preceding sections we have considered a model for a thermostat with the left end ($x=0$) insulated. In the following more realistic model we consider the leakage of heat from left end. Thus the initial-boundary value problem for the temperature $u(x,t)$ is as follows:

$$\begin{aligned}u_t &= u_{xx}, && 0<x<1,\ t>0, && (7.1a)\\ u(x,0) &= \Phi(x), && 0<x<1, && (7.1b)\\ u_x(0,t) &= \beta(u(0,t)+1), && t>0, && (7.1c)\\ u_x(1,t) &= -\gamma u(0,t), && t>0, && (7.1d)\end{aligned}$$

where $\beta>0$ and $\gamma>0$. Here we consider the linearized problem directly. Strictly we should start with the nonlinear problem (7.1) (i.e. with (7.1d) replaced by $u_x(1,t)=H(u(0,t))$), perform a semi-discretisation, a linearisation and then the determination of the eigenvalues of the resulting spatially discrete operator or matrix. However, again we can show that the eigenvalues tend to the poles associated with the linearized problem. Thus we proceed directly to the linear problem, ob-

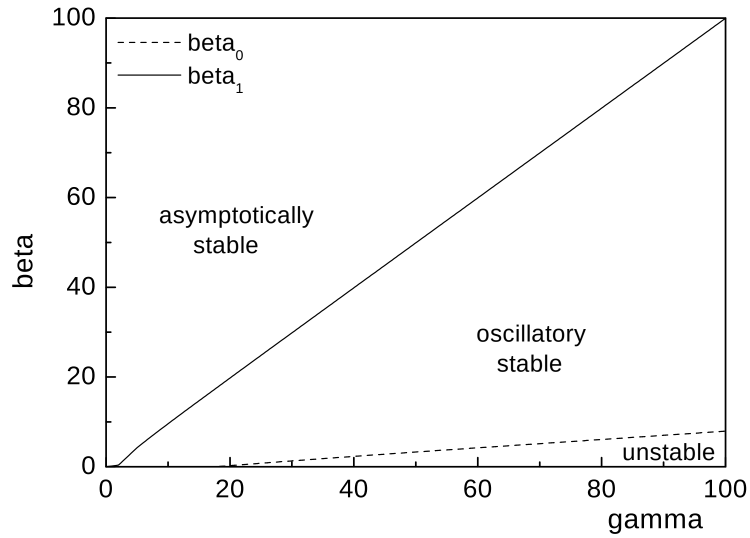


Figure 4: Stability diagram: relation between γ and β.

taining the solution

$$
\begin{aligned}
u(x,t) \quad = \quad & \sum_{\forall n} \frac{2e^{z_n t}\left(\sqrt{z_n}\cosh(\sqrt{z_n}x) + \beta\sinh(\sqrt{z_n}x)\right)}{(1+\beta)\sinh\sqrt{z_n} + \sqrt{z_n}\cosh\sqrt{z_n}} \int_0^1 \Phi(\xi)\cosh(\sqrt{z_n}(1-\xi))\, d\xi \\
& - \sum_{\forall n} \frac{2e^{z_n t}\beta\cosh\sqrt{z_n}\left(\sqrt{z_n}\cosh(\sqrt{z_n}x) + \beta\sinh(\sqrt{z_n}x)\right)}{z_n\left((1+\beta)\sinh\sqrt{z_n} + \sqrt{z_n}\cosh\sqrt{z_n}\right)} \\
& + \frac{\beta}{\beta+\gamma}(\gamma x - 1).
\end{aligned}
$$

A further sum of residues corresponding to double poles will need to be added for appropriate values of γ and β.

The presence of zero poles results in critical stability of the solution (7.2). It means that if this solution is asymptotically or oscillatory stable, it does not tend to zero as observed in §4. Indeed, as t tends to infinity the sums in (7.2) tend to zero, and therefore $u(x, t_\infty)$ tends to $\beta(\gamma x - 1)/(\beta + \gamma)$.

Once again the same pattern has been observed. The particular character of the solution is governed by two parameters γ and β; figure 4 displays a $\beta - \gamma$ stability diagram.

8. A Model with Heat Loss

The partial differential equation describing the models in the preceding sections is homogeneous. Let us now consider a model for a thermostat where leakage along the heating system is allowed. Then the temperature distribution is given by the solution of the initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx} - \omega u, & 0 < x < 1,\ t > 0, \\ u(x,0) &= \Phi(x), & 0 < x < 1, \\ u_x(0,t) &= 0, & t > 0, \\ u_x(1,t) &= -\gamma u(0,t), & t > 0, \end{aligned}$$

where $\omega > 0$ and $\gamma > 0$. Using the transformation

$$u(x,t) = e^{-\omega t} v(x,t)$$

it is not difficult to show that the solution $u(x,t)$ can be written as follows

$$u(x,t) = \sum_{\forall n} \frac{2e^{(z_n - \omega)t}\sqrt{z_n}\cosh(\sqrt{z_n}x)}{\sinh\sqrt{z_n} + \sqrt{z_n}\cosh\sqrt{z_n}} \int_0^1 \Phi(\xi)\cosh(\sqrt{z_n}(1-\xi))\, d\xi.$$

We can show that even when the original problem is unstable (i.e. $\gamma > 17.798542$), the parameter ω can make the solution stable.

Figure 5 is a stability diagram displaying three regions of stability within the (γ, ω)-plane: exponential stability, oscillatory but exponentially damped behaviour and instability. It is noted that the poles with positive real part may be translated back into the oscillatory stable region by choosing ω sufficiently large, but never into the exponentially stable region. An open question remains: is there an analytic expression for the dividing stability curve.

9. Concluding Remarks

This paper has considered a paradigm for the air-conditioning problem where the controller or thermostat is situated at some distance from the extractor unit. This paradigm consisted of a one-dimensional heat conduction equation with a nonlocal and nonlinear boundary condition. Existence and uniqueness were first established through its equivalent representation as a Volterra integral equation.

Semi-discretisation of the spatial variable resulted in a nonlinear system of ordinary differential equations whose stability was studied through linearisation and application of the Hopf bifurcation theorem. By considering the poles associated with the linearized problem an analytic solution was written down and its numerical evaluation was discussed. Further it was shown that the eigenvalues associated with the discrete linear operator tended, in the limit as the dimensionality increased, to the poles and this fact was used to understand the stability of the solution of the

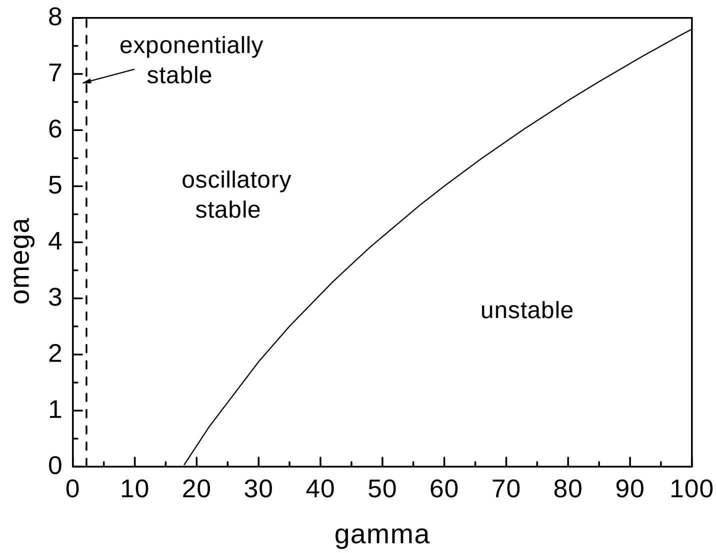


Figure 5: Stability diagram: relation between γ and ω.

original nonlinear problem. Indeed the solution was shown to be asymptotically stable, oscillatory but bounded by a negative exponential, or oscillatory and unbounded for different ranges of a parameter characterising the heat transfer coefficient. An almost local problem was considered whereby the thermostat was situated close to the extractor unit. It was shown that an unstable solution could still exist: however the closer the thermostat was to the extractor unit, the larger the value of γ needed to be to cause instability. The more practical cases when flux leakage from one end or heat loss were permitted were also studied with similar results: the larger the heat transfer coefficient the larger the value of γ required to cause instability.

The practical conclusions are that air-conditioning systems can be unstable, that the position of the thermostat might well be crucial, and stability or otherwise certainly depends upon the power input. In colder climates central heating might be of more interest. In this case the boiler is either switched on or off according to the temperature where the controller is situated. It would appear that whether this problem with flux leakage is stable or not is still an open question.

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