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Simultaneous Controllability for a System with Resistance Term

G.O. ANTUNES, F.A. ARARUNA, L.A. MEDEIROS[1], Instituto de Matemática, UFRJ, Caixa Postal 68530, 21945-970 Rio de Janeiro, RJ, Brasil.

Abstract. In this work we study the simultaneous controllability for a system of equations that constitutes a model of dynamical elasticity for incompressible materials.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with regular boundary Γ. Let Q denote the cylinder $\Omega \times (0,T)$ whose lateral boundary is given by $\Sigma = \Gamma \times (0,T)$.

In this work, we shall consider the simultaneous controllability for the system

$$\left| \begin{array}{lll} y_1^{''} - \Delta y_1 = -\nabla p & \text{in} & Q \\ y_2^{''} - \Delta y_2 = -\nabla q & \text{in} & Q \\ div\ y_1 = 0 & \text{in} & Q \\ div\ y_2 = 0 & \text{in} & Q \\ y_1 = v & \text{on} & \Sigma \\ \dfrac{\partial y_2}{\partial \nu} = w & \text{on} & \Sigma \\ y_1(0) = y_1^0,\ y_1'(0) = y_1^1 & \text{in} & \Omega \\ y_2(0) = y_2^0,\ y_2'(0) = y_2^1 & \text{in} & \Omega \end{array} \right. \tag{1.1}$$

where $p = p(x,t)$ and $q = q(x,t)$ denote the resistance terms.

Physically the above system models the small deformations or displacements of the solid body $\Omega \subset \mathbb{R}^n$ composed of incompressible elastic materials, subject to controls acting on the boundary Σ.

The simultaneous controllability for the system (1.1) is formulated as follows: given $T > 0$ large enough, find a Hilbert space H such that for every set $\{y_1^0, y_1^1, y_2^0, y_2^1\}$ belonging to H, there exists a pair of controls $\{v, w\}$, such that a solution $\{y_1(v), y_2(w)\}$ of (1.1) satisfies the equilibrium condition

$$y_1(T) = y_1'(T) = y_2(T) = y_2'(T) = 0, \tag{1.2}$$

and

$$w = \frac{\partial v}{\partial t} \quad \text{on} \quad \Sigma.$$

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We investigate this problem by means of the Hilbert Uniqueness Method (HUM) idealized by Lions [6].

The problem of the simultaneous controllability was initially studied by Lions [6]. Kapitonov [4] investigated a similar question. For exact controllability we mention Cavalcanti et al [2].

2. Notations, Assumptions and Results

We consider $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, $n \geq 2$, two bounded domains with boundary $\partial\Omega_0, \partial\Omega_1$ of class C^2, such that

$$\overline{\Omega}_1 \subset \Omega_0 \ , \tag{2.1}$$

$$\Omega_0,\ \Omega_1 \text{ are star shaped with respect to } x_0 \in \overline{\Omega}_1. \tag{2.2}$$

Let us assume

$$\Omega = \Omega_0 \setminus \overline{\Omega}_1. \tag{2.3}$$

We set $m(x) = x - x_0$, $R(x_0) = \max\limits_{x \in \overline{\Omega}} |m(x)|$ and define

$$\Gamma(x_0) = \{x \in \Gamma; m(x) \cdot \nu(x) > 0\} \qquad \text{and} \qquad \Gamma_*(x_0) = \Gamma \setminus \Gamma(x_0).$$

The following partition of the boundary is chosen

$$\Gamma_0 = \Gamma(x_0), \quad \Gamma_1 = \Gamma_*(x_0).$$

The action in the boundary Σ is assumed to be of the following type

$$y_1 = \begin{cases} v & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,T) \\ 0 & \text{on} \quad \Sigma \diagdown \Sigma_0 = \Gamma_1 \times (0,T) \end{cases},$$

$$\begin{aligned} &\frac{\partial y_2}{\partial \nu} = w \quad \text{on} \quad \Sigma_0, \\ &y_2 = 0 \qquad \text{on} \quad \Sigma \diagdown \Sigma_0. \end{aligned}$$

In addition we consider the following hypotheses

$$p = q = 0 \quad \text{on} \quad \Sigma_0. \tag{2.4}$$

We introduce the following Hilbert spaces

$$V = \left\{ u \in \left(H_0^1(\Omega)\right)^n ; \, div \ u = 0 \right\},$$

$$H = \left\{ u \in \left(L^2(\Omega)\right)^n ; \, div \ u = 0, \ u \cdot \eta = 0 \text{ on } \Gamma \right\},$$

with the structure of internal product and norm induced by $\left(H_0^1(\Omega)\right)^n$ and $\left(L^2(\Omega)\right)^n$, respectively. We still consider

$$\mathcal{V} = \left\{ \varphi \in (D(\Omega))^n ; \ div \ \varphi = 0 \right\},$$

$$X = \left\{ \varphi \in \left(H^1(\Omega)\right)^n ; \, div \; \varphi = 0, \; \varphi = 0 \text{ on } \Gamma_1 \right\},$$

and

$$Y = \left\{ \varphi \in X; \; \Delta\varphi \in \left(L^2(\Omega)\right)^n, \, \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}.$$

The energy associated with the system (1.1) is given by

$$E(t) = E_1(t) + E_2(t), \tag{2.5}$$

where

$$E_i(t) = \frac{1}{2} \left\{ \sum_{k=1}^{n} \int_\Omega |\nabla y_{ki}(t)|^2 \, dx + \sum_{k=1}^{n} \int_\Omega |y'_{ki}(t)|^2 \, dx \right\}, \quad i = 1, 2. \tag{2.6}$$

3. Inverse Inequality

Let us consider the following problem

$$\left| \begin{array}{lll} \Phi_1'' - \Delta\Phi_1 = -\nabla p & \text{in} & Q \\ div \; \Phi_1 = 0 & \text{in} & Q \\ \Phi_1 = 0 & \text{on} & \Sigma \\ \Phi_1(0) = \Phi_1^0 \, , \Phi_1'(0) = \Phi_1^1 & \text{in} & \Omega. \end{array} \right. \tag{3.1}$$

Lions in [7] showed that the solution Φ_1 of (3.1) has the hidden regularity $\dfrac{\partial u}{\partial \nu} \in \left(L^2(\Sigma)\right)^n$ and that mapping

$$\left\{\Phi_1^0, \Phi_1^1\right\} \mapsto \frac{\partial u}{\partial \nu} \tag{3.2}$$

is continuous from $V \times H$ in $\left(L^2(\Sigma)\right)^n$.

Remark 3.1: Multiplying the equation in $(3.1)_1$ by $m\nabla\Phi_1$ and integrating in Q,

$$\int_Q \Phi_1'' m \nabla y_1 dxdt - \int_Q \Delta\Phi_1 m \nabla \Phi_1 dxdt = \int_Q (-\nabla p) \, m \nabla \Phi_1 dxdt.$$

Let us put

$$X = -\int_Q \frac{\partial p}{\partial x_i} m_k \frac{\partial \Phi_{1i}}{\partial x_k} dxdt,$$

with the summation convention of repeated indices. Integrating by parts in x_k and observing $(3.1)_3$ comes that

$$\begin{aligned} X &= -\int_0^T \frac{\partial p}{\partial x_i} m_k \Phi_{1i} dt \Big|_\Gamma + \int_Q \frac{\partial}{\partial x_k} \left(\frac{\partial p}{\partial x_i} m_k \right) \Phi_{1i} dxdt \\ &= \int_Q \frac{\partial^2 p}{\partial x_k \partial x_i} m_k \Phi_{1i} dxdt + \int_Q \frac{\partial p}{\partial x_i} \frac{\partial m_k}{\partial x_k} \Phi_{1i} dxdt \\ &= \int_Q \frac{\partial^2 p}{\partial x_k \partial x_i} m_k \Phi_{1i} dxdt + n \int_Q \frac{\partial p}{\partial x_i} \Phi_{1i} dxdt. \end{aligned}$$

Now,

$$n\int_Q \frac{\partial p}{\partial x_i}\Phi_{1i}dxdt = n\int_0^T p\Phi_{1i}dt|_\Gamma - n\int_Q p\frac{\partial \Phi_{1i}}{\partial x_i}dxdt = -n\int_Q p\ div\ \Phi_1 dxdt = 0.$$

Therefore

$$X = \int_Q \frac{\partial^2 p}{\partial x_k \partial x_i} m_k \Phi_{1i} dxdt.$$

Making integration by parts again, it results in

$$\begin{aligned} X &= \int_0^T \frac{\partial p}{\partial x_k} m_k \Phi_{1i} dt\bigg|_\Gamma - \int_Q \frac{\partial p}{\partial x_k}\frac{\partial}{\partial x_i}\left(m_k \Phi_{1i}\right) dxdt \\ &= -\int_Q \frac{\partial p}{\partial x_k}\frac{\partial m_k}{\partial x_i}\Phi_{1i}dxdt - \int_Q \frac{\partial p}{\partial x_k} m_k \frac{\partial \Phi_{1i}}{\partial x_i} dxdt \\ &= -\int_Q \frac{\partial p}{\partial x_k}\delta_i^k \Phi_{1i} dxdt - \int_Q \frac{\partial p}{\partial x_k} m_k\ div\Phi_1 dxdt \\ &= \int_Q p\ div\Phi_1 dxdt - \int_Q \frac{\partial p}{\partial x_k} m_k\ div \Phi_1 dxdt = 0. \end{aligned}$$

Lemma 3.1. *Assume* $q = q(x) \in \left[C^1\left(\overline{\Omega}\right)\right]^n$*. Then, for every solution of* (3.1) *with data* $\left\{\Phi_1^0, \Phi_1^1\right\} \in V \times H$*, the following identity holds:*

$$\begin{aligned} &\frac{1}{2}\sum_{k=1}^n \int_\Sigma q_k(x)\cdot\nu_k(x)\left|\frac{\partial \Phi_1}{\partial \nu}\right|^2 d\Sigma \\ &= \left(\Phi_1'(t), q(x)\nabla\Phi_1\right)\big|_0^T dx + \sum_{k=1}^n \int_Q \frac{\partial p}{\partial x_i} q_k \frac{\partial \Phi_{1i}}{\partial x_k} dxdt \\ &\quad + \frac{1}{2}\sum_{k=1}^n \int_Q \frac{\partial q_k}{\partial x_k}\left(\left|\Phi_1'(x,t)\right|^2 - \left|\nabla\Phi_1(x,t)\right|^2\right) dxdt + \sum_{j,k=1}^n \int_Q \frac{\partial q_k}{\partial x_j}\frac{\partial \Phi_{1i}}{\partial x_k}\frac{\partial \Phi_{1i}}{\partial x_j} dxdt. \end{aligned}$$

Lemma 3.2. *Assume* $T > 2R(x_0)$*. Then the following estimate holds for every solution of* (3.1) *with data* $\left\{\Phi_1^0, \Phi_1^1\right\} \in V \times H$*,*

$$E_{01} \leq \frac{R(x_0)}{2\left(T - 2R(x_0)\right)}\int_{\Sigma_0}\left|\frac{\partial \Phi_1}{\partial \nu}\right|^2 d\Sigma. \tag{3.3}$$

In the proof of the Lemmas 3.1 and 3.2 we used the idea of Lions [6] together with the Remark 3.1.

We consider now the homogeneous problem for Φ_2

$$\left| \begin{array}{lll} \Phi_2'' - \Delta\Phi_2 = -\nabla q & \text{in} & Q \\ div\ \Phi_2 = 0 & \text{in} & Q \\ \dfrac{\partial \Phi_2}{\partial \nu} = 0 & \text{on} & \Sigma_0 \\ \Phi_2 = 0 & \text{on} & \Sigma_1 \\ \Phi_2(0) = \Phi_2^0,\ \Phi_2'(0) = \Phi_2^1 & \text{in} & \Omega. \end{array} \right. \tag{3.4}$$

Similarly to the Remark 3.1, multiplying the equation in $(3.4)_1$ by $m\nabla\Phi_2$ and integrating in Q, we obtain

$$\int_Q \Phi_2'' m\nabla\Phi_2 dxdt - \int_Q \Delta\Phi_2 m\nabla\Phi_2 dxdt = \int_Q (-\nabla q)\, m\nabla\Phi_2 dxdt = 0. \tag{3.5}$$

Lemma 3.3. *Assume* $q \in \left[W^{1,\infty}(\Omega)\right]^n$. *Then for every weak solution of the homogeneous problem* (3.4) *with data* $\left\{\Phi_2^0, \Phi_2^1\right\} \in Y \times X$ *the following identity holds*

$$\begin{aligned}\int_{\Sigma_0} q_k\nu_k \left(\left|\Phi_2'\right|^2 \right. &- \left. \left|\nabla_\sigma \Phi_2\right|^2\right) d\Sigma + \frac{1}{2}\int_{\Sigma_1} q_k\nu_k \left|\frac{\partial\Phi_2}{\partial\nu}\right|^2 d\Sigma \\ &= \left(\Phi_{2i}'(t), q_k \frac{\partial\Phi_{2i}(t)}{\partial x_k}\right)\Bigg|_0^T + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\left(\left|\Phi_2'\right|^2 - \left|\nabla\Phi_2\right|^2\right) dxdt \\ &\quad + \int_Q \frac{\partial q_k}{\partial x_j}\frac{\partial \Phi_{2i}}{\partial x_j}\frac{\partial\Phi_{2i}}{\partial x_k} dxdt,\end{aligned}$$

where $\nabla_\sigma\Phi_2$ *denotes the tangential gradient of* Φ_2.

Let $\lambda_0^2 > 0$ be the first eigenvalues of the following spectral problem

$$\left|\begin{array}{lll} -\Delta\Phi = \lambda^2\Phi & \text{in} & \Omega \\ div\ \Phi = 0 & \text{in} & \Omega \\ \dfrac{\partial\Phi}{\partial\nu} = 0 & \text{on} & \Gamma_0 \\ \Phi = 0 & \text{on} & \Gamma_1. \end{array}\right. \tag{3.6}$$

Lemma 3.4. *Assume* $T > 2R(x_0) + \dfrac{(n-1)}{\lambda_0}$. *Then every solution of the* (3.4) *with data* $\left\{\Phi_2^0, \Phi_2^1\right\} \in Y \times X$ *verifies*

$$E_{02} \leq \frac{R(x_0)}{2\left(T - 2R(x_0) - \dfrac{n-1}{\lambda_0}\right)} \int_{\Sigma_0} \left|\Phi_2'\right|^2 d\Sigma. \tag{3.7}$$

The proof of the Lemmas 3.3 and 3.4 were done as in Lions [6] adapted to (3.5).

Theorem 3.1. *Let* Ω *be a domain satisfying* (2.1) − (2.3) *and* $T > 4R(x_0) + \dfrac{n-1}{\lambda_0}$. *Then for every data,*

$$\left\{\Phi_1^0, \Phi_1^1\right\} \in V \times H, \quad \left\{\Phi_2^0, \Phi_2^1\right\} \in Y \times X,$$

the solutions Φ_1, Φ_2 *of the homogeneous system*

$$\left|\begin{array}{lll} \Phi_1'' - \Delta\Phi_1 = -\nabla p & in & Q \\ \Phi_2'' - \Delta\Phi_2 = -\nabla q & in & Q \\ div\ \Phi_1 = 0 & in & Q \\ div\ \Phi_2 = 0 & in & Q \\ \Phi_1 = 0 & on & \Sigma \\ \dfrac{\partial\Phi_2}{\partial\nu} = 0 & on & \Sigma_0 \\ \Phi_2 = 0 & on & \Sigma_1 \\ \Phi_1(0) = \Phi_1^0,\ \Phi_1'(0) = \Phi_1^1 & in & \Omega \\ \Phi_2(0) = \Phi_2^0,\ \Phi_2'(0) = \Phi_2^1 & in & \Omega \end{array}\right. \tag{3.8}$$

verify

$$E_0 \leq \frac{R(x_0)}{2(T - T(x_0))} \int_{\Sigma_0} \left(\frac{\partial\Phi_1}{\partial\nu} + \Phi_2'\right)^2 d\Sigma. \tag{3.9}$$

Proof. From (3.3) and (3.7) we obtain for $T > 2R(x_0) + \dfrac{n-1}{\lambda_0}$,

$$E_0 \leq \frac{R(x_0)}{2\left(T - 2R(x_0) - \dfrac{n-1}{\lambda_0}\right)} \int_{\Sigma_0} \left(\left|\frac{\partial\Phi_1}{\partial\nu}\right|^2 + |\Phi_2'|^2\right) d\Sigma. \tag{3.10}$$

Suppose that the following inequality is verified

$$\left|\int_{\Sigma_0} \frac{\partial\Phi_1}{\partial\nu}\Phi_2' d\Sigma\right| \leq 2E_0. \tag{3.11}$$

Then from (3.10) and (3.11) it follows

$$E_0 \leq \frac{R(x_0)}{2\left(T - 4R(x_0) - \dfrac{n-1}{\lambda_0}\right)} \int_{\Sigma_0} \left(\frac{\partial\Phi_1}{\partial\nu} + \Phi_2'\right)^2 d\Sigma, \tag{3.12}$$

for $T > 4R(x_0) + \frac{n-1}{\lambda_0}$.

To conclude the proof of the theorem, it remains then to verify that (3.11) happens.

In fact, multiplying the equation in $(3.8)_1$ by Φ_2', and integrating on Q, we obtain

$$\int_Q (\Phi_1''\Phi_2' + \nabla\Phi_1\nabla\Phi_2')\,dxdt - \int_\Sigma \frac{\partial\Phi_1}{\partial\nu}\Phi_2' d\Sigma = \int_Q p\,div\ \Phi_2' dxdt - \int_\Sigma p\Phi_2' d\Sigma. \tag{3.13}$$

Since $\Phi_2 \in C([0,T], Y) \cap C^1([0,T], X)$, p satisfies (2.4) and $\Phi_2 = 0$ on Σ_1, then from (3.13), it follows

$$\int_Q (\Phi_1''\Phi_2' + \nabla\Phi_1\nabla\Phi_2')\,dxdt = \int_{\Sigma_0} \frac{\partial\Phi_1}{\partial\nu}\Phi_2' d\Sigma. \tag{3.14}$$

Now, multiplying the equation $(3.8)_2$ by Φ_1', and integrating in Q, we get

$$\int_Q \left(\Phi_2''\Phi_1' + \nabla\Phi_2\nabla\Phi_1'\right) dxdt = \int_Q q\ div\ \Phi_1' dxdt - \int_\Sigma q\Phi_1' d\Sigma.$$

Since $\Phi_1 \in C\left([0,T], V\right) \cap C^1\left([0,T], H\right)$, then

$$\int_Q \left(\Phi_2''\Phi_1' + \nabla\Phi_2\nabla\Phi_1'\right) dxdt = 0. \tag{3.15}$$

Adding (3.14) and (3.15) comes

$$\int_Q \frac{d}{dt}\left(\Phi_1'\Phi_2' + \nabla\Phi_1\nabla\Phi_2\right) dtdx = \int_{\Sigma_0} \frac{\partial\Phi_1}{\partial\nu}\Phi_2' d\Sigma,$$

that is,

$$\left\{\left(\Phi_1'(t), \Phi_2'(t)\right) + \left(\nabla\Phi_1(t), \nabla\Phi_2(t)\right)\right\}\Big|_0^T = \int_{\Sigma_0} \frac{\partial\Phi_1}{\partial\nu}\Phi_2' d\Sigma.$$

Therefore,

$$\left|\int_{\Sigma_0} \frac{\partial\Phi_1}{\partial\nu}\Phi_2' d\Sigma\right| \le 2\left(E_{01} + E_{02}\right) = 2E_0,$$

concluding the result. ■

Corollary 3.1. *Assume* Ω *as in the Theorem* 3.1 and $T > 4R(x_0) + \dfrac{n-1}{\lambda_0}$. Let Φ_1 and Φ_2 be two solutions corresponding to the initial data $\left\{\Phi_1^0, \Phi_1^1\right\} \in V \times H$ and $\left\{\Phi_2^0, \Phi_2^1\right\} \in Y \times X$ respectively. If Φ_1 and Φ_2 satisfy

$$\frac{\partial\Phi_1}{\partial\nu} + \Phi_2' = 0 \quad \text{on} \quad \Sigma_0, \qquad \text{then} \qquad \Phi_1 = \Phi_2 = 0 \quad \text{in} \quad Q.$$

Proof. The proof follows immediately from (3.9). ■

4. Simultaneous Controllability

The main result of this work is the following theorem:

Theorem 4.2. *Let* Ω *be a bounded domain of the* $\mathbb{R}^n$, $n \ge 2$, *satisfying* $(2.1)-(2.3)$ *and* $T > 4R(x_0) + \dfrac{n-1}{\lambda_0}$. *Then for every data* $\left\{y_1^0, y_1^1, y_2^0, y_2^1\right\} \in H' \times V' \times \left(L^2(\Omega)\right)^n \times X'$, *there exists a control* $v \in \left(L^2(\Sigma_0)\right)^n$, *such that the solution* $\{y_1, y_2\}$ *of the system*

$$\left| \begin{array}{lll} y_1'' - \Delta y_1 = -\nabla p & in & Q \\ y_2'' - \Delta y_2 = -\nabla q & in & Q \\ div\ y_1 = div\ y_2 = 0 & in & Q \\ y_1 = \begin{cases} v & on \quad \Sigma_0 \\ 0 & on \quad \Sigma_1 \end{cases} & & \\ \dfrac{\partial y_2}{\partial \nu} = \dfrac{\partial v}{\partial t} & on & \Sigma_0 \\ y_2 = 0 & on & \Sigma_1 \\ y_1(0) = y_1^0,\ y_1'(0) = y_1^1 & in & \Omega \\ y_2(0) = y_2^0,\ y_2'(0) = y_2^1 & in & \Omega \end{array} \right. \tag{4.1}$$

verifies

$$y_1(T) = y_1'(T) = y_2(T) = y_2'(T) = 0.$$

Proof. We will apply here the HUM.

First we solve the homogeneous system (3.8) with the initial conditions $\left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\} \in \mathcal{V} \times \mathcal{V} \times Y \times X$.

Let us define the quadratic form

$$\left\| \left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\} \right\|_F := \left\{ \int_{\Sigma_0} \left| \frac{\partial \Phi_1}{\partial \nu} + \Phi_2' \right|^2 \right\}^{\frac{1}{2}}. \tag{4.2}$$

It follows from the Corollary 3.1, that (4.2) defines a norm in $\mathcal{V} \times \mathcal{V} \times Y \times X$.

We build the space,

$$F = \overline{\mathcal{V} \times \mathcal{V} \times Y \times X}^{\|\cdot\|_F}.$$

From (3.12) follows the immersion

$$F \hookrightarrow V \times H \times X \times \left(L^2(\Omega)\right)^n. \tag{4.3}$$

Therefore

$$V' \times H' \times X' \times \left(L^2(\Omega)\right)^n \hookrightarrow F'$$

with continuous immersion.

Note that

$$\left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\} \in F \Leftrightarrow \frac{\partial \Phi_1}{\partial \nu} + \Phi_2' \in \left(L^2(\Sigma_0)\right)^n. \tag{4.4}$$

On the other hand, from (3.3), the continuity of the application (3.2) and (4.3), we obtain

$$\left\{\Phi_1^0, \Phi_1^1\right\} \in V \times H \Leftrightarrow \frac{\partial \Phi_1}{\partial \nu} \in \left(L^2(\Sigma_0)\right)^2. \tag{4.5}$$

Hence, from (4.4) and (4.5), it follows that

$$\left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\} \in F \Leftrightarrow \begin{cases} \left\{\Phi_1^0,\ \Phi_1^1\right\} \in V \times H \\ \text{and} \\ \Phi_2' \in \left(L^2(\Sigma_0)\right)^n. \end{cases}$$

Thus, we consider the norm

$$\left\| \left\{ \Phi_2^0, \Phi_2^1 \right\} \right\|_G := \left\{ \int_{\Sigma_0} \left| \Phi_2' \right|^2 d\Sigma \right\}^{\frac{1}{2}},$$

and the Hilbert space

$$G = \overline{Y \times X}^{\|\cdot\|_G}.$$

From (3.7) , it follows

$$G \hookrightarrow X \times \left(L^2 \left(\Omega \right) \right)^n.$$

Therefore,

$$F = V \times H \times G$$

and

$$F' = V' \times H' \times G'.$$

We consider the following backward system

$$\left| \begin{array}{lll} \Psi_1'' - \Delta \Psi_1 = -\nabla p & \text{in} & Q \\ \Psi_2'' - \Delta \Psi_2 = -\nabla q & \text{in} & Q \\ div\ \Psi_1 = div\ \Psi_2 = 0 & \text{in} & Q \\ \Psi_1 = \dfrac{\partial \Phi_1}{\partial \nu} + \Phi_2' & \text{on} & \Sigma_0 \\ \Psi_1 = 0 & \text{on} & \Sigma_1 \\ \dfrac{\partial \Psi_2}{\partial \nu} = \dfrac{\partial}{\partial t} \left(\dfrac{\partial \Phi_1}{\partial \nu} + \Phi_2' \right) & \text{on} & \Sigma_0 \\ \Psi_2 = 0 & \text{on} & \Sigma_1 \\ \Psi_1 \left(T \right) = \Psi_1' \left(T \right) = \Psi_2 \left(T \right) = \Psi_2' \left(T \right) = 0 & \text{in} & \Omega, \end{array} \right. \tag{4.6}$$

where $\dfrac{\partial}{\partial t} \left(\dfrac{\partial \Phi_1}{\partial \nu} + \Phi_2' \right)$ is taken in the following sense

$$\left\langle \frac{\partial}{\partial t} \left(\frac{\partial \Phi_1}{\partial \nu} + \Phi_2' \right), v \right\rangle = - \int_{\Sigma_0} \left(\frac{\partial \Phi_1}{\partial \nu} + \Phi_2' \right) v' d\Sigma,$$

for all $v \in H_0^1 \left(0, T; \left(L^2 \left(\Gamma_0 \right) \right)^n \right)$.

Consider now the application

$$\Lambda : F \to F',$$

defined by

$$\Lambda \left\{ \Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1 \right\} = \left\{ \Psi_1' \left(0 \right), -\Psi_1 \left(0 \right), \Psi_2' \left(0 \right), -\Psi_2 \left(0 \right) \right\}, \tag{4.7}$$

where $\{\Psi_1, \Psi_2\}$ is the solution of (4.6).

The norm in (4.2) induces in $\mathcal{V} \times \mathcal{V} \times Y \times X$ the following inner product

$$\left\langle \left\{ \Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1 \right\}, \left\{ \xi_1^0, \xi_1^1, \xi_2^0, \xi_2^1 \right\} \right\rangle_F = \int_{\Sigma_0} \left(\frac{\partial \Phi_1}{\partial \nu} + \Phi_2' \right) \left(\frac{\partial \xi_1}{\partial \nu} + \xi_2' \right) d\Sigma,$$

hence

$$\left\langle \Lambda \left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\}, \left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\}\right\rangle_F = \left\|\left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\}\right\|_F^2$$

and Λ is a isomorphism between F and F'. Therefore, for every $\left\{y_1^0, y_1^1, y_2^0, y_2^1\right\} \in F'$, there exists only one $\left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\} \in F$ such that

$$\Lambda \left\{\Phi_1^0, \Phi_1^1, \Phi_2^0, \Phi_2^1\right\} = \left\{y_1^1, -y_1^0, y_2^1, -y_2^0\right\}. \tag{4.8}$$

By (4.7) and (4.8) we conclude that the unique ultra weak solution of (4.6) satisfies (1.1). Then the unique ultra weak solution of (1.1), with control $v = \dfrac{\partial \Phi_1}{\partial \nu} + \Phi_2'$ verifies (1.2). ■

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Resumo. Neste trabalho estudamos a controlabilidade simultânea para um sistema de equações que representam um modelo da dinâmica de elasticidade para materiais imcompressíveis.

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