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The Non-Homogeneous Wave Equation in Non-Cylindrical Domains

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Abstract. We prove the existence and uniqueness of a global regular solution to the mixed problem for the nonhomogeneous wave equation in a noncylindrical domain using a hyperbolic nonsingular smooth transformation of the original problem to a nonlinear hyperbolic problem in a cylinder.

1. Introduction

Solvability of various initial boundary value problems for linear and nonlinear hyperbolic equations in noncylindrical domains was studied by many authors [1, 2, 5−11]. These problems are interesting and non-obvious from the very beginning-formulation of a problem: what kind of conditions to impose at lateral (moving) boundaries of a domain. The biggest part of the papers in question was devoted to the mixed problem with the Dirichlet condition at the lateral boundary. In our paper we formulate the mixed problem with nonlinear first order differential operator at the moving boundary. In cylindrical domains this conditions was used by various authors [3, 4]. It is of the form:

$$\left.\frac{\partial u}{\partial \nu} + \varphi(u_t)\right|_{\Gamma} = 0,$$

where $\frac{\partial u}{\partial \nu}$ is the derivative in the direction of the exterior normal on the boundary Γ, u_t is the tangential derivative on Γ.

In our paper, we propose the condition of the form:

$$\left.\frac{\partial u}{\partial \nu} + |u_s|^{\rho}\, u_s\right|_{\Gamma(t)} = 0,$$

where u_s is the tangential derivative at the moving boundary $\Gamma(t)$.

We prove the existence and uniqueness of a global regular solution to the mixed problem in a noncylindrical domain using a hyperbolic nonsingular smooth transformation of the original problem to a nonlinear hyperbolic problem in a cylinder. Then we exploit the Galerkin method to prove the existence result. The inverse transformation gives the desired result.

[1]Partially supported by CNPq.
[2]Supported by CNPq

2. Notations and the Main Results

Let Q_T be a non cylindrical domain of $\mathbf{R^2}$ defined by:

$$Q_T = \{(x,t) \in \mathbf{R^2}; \alpha_1(t) < x < \alpha_2(t), 0 \le t < T\},$$

$$D_t = Q_T \cap (\tau = t),$$

where $\alpha_1(t), \alpha_2(t) \in C^2[0,T)$. We consider the following problem:

$$u_{tt} - k^2 u_{xx} = f(x,t) \qquad \text{in} \qquad Q_T, \tag{2.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \alpha_1(0) < x < \alpha_2(0), \tag{2.2}$$

$$u\Big|_{x=\alpha_2(t)} = 0, \qquad u_x + \beta \left|u_s\right|^\rho u_s\Big|_{x=\alpha_1(t)} = 0, \qquad t > 0, \tag{2.3}$$

where $u_s = u_t + \alpha_1'(t)u_x$, $\rho \ge 0$, β, k are given constants. We denote by $V(D_t)$ the following space of functions:

$$V(D_t) = \{g \in H^1(D_t), \quad g(\alpha_2(t), t) = 0\},$$

$$\gamma(t) = \alpha_2(t) - \alpha_1(t),$$

$$(u,v)(t) = \int_{D_t} u(x,t)v(x,t)dx.$$

Theorem 1 *Let $\beta < 0$, $\alpha_1'(t) \ge 0$, $u_0 \in H^2(D_0) \cap V(D_0), u_1 \in V(D_0)$ and the following conditions hold:*

(i) $0 < \gamma_0 \le \gamma(t) \le \gamma_1 < \infty, \quad \forall t > 0;$

(ii) $u_0'(0) + \beta\gamma(0) \left|v_1(0)\right|^\rho v_1(0) = 0,$ *where* $v_1(0) = u_1(\alpha_1(0)) + \alpha_2' u_0'(\alpha_1(0));$

(iii) $|y\alpha_2'(t) + (1-y)\alpha_1'(t)| \le k_1 k, \ k_1 \in (0,1); \forall y \in [0,1].$

Then for any $f \in H^1(0,T;L^2(D_t))$ there exists a unique strong solution to $(2.1) - (2.3):$

$$u \in L^\infty(0,T; H^2(D_t) \cap V(D_t));$$
$$u_t \in L^\infty(0,T; V(D_t));$$
$$u_{tt} \in L^\infty(0,T; L^2(D_t)).$$

Proof: To prove this result, we transform Q_Y in a cylinder $Q = (0,1) \times (0,T)$ by the transformation,

$$y = \frac{x - \alpha_1(t)}{\alpha_2(t) - \alpha_1(t)}, \qquad \tau = t, \qquad v(y,\tau) \equiv u(x(y,\tau),\tau). \tag{2.4}$$

In these variables, with τ replaced by t, the problem $(1.1) - (1.3)$ is reduced to the form:

$$v_{tt} - a_1 v_{yy} - 2a_2 v_{yt} - b_1 v_y = f_1(y,t) \quad \text{in} \quad Q, \tag{2.5}$$

$$\begin{aligned} &v(y,0) = u_0(x(y,0),0) = v_0(y) \quad \text{in} \quad (0,1), \\ &v_t(y,0) = u_1(x(y,0),0) + \alpha_1'(0)\gamma(0) \\ &\qquad + \tfrac{\alpha_1(0)\gamma'(0))}{\gamma(0)} u_0'(x(y,0)) = v_1(y), \quad \text{in} \quad (0,1), \end{aligned} \tag{2.6}$$

$$\begin{aligned} &v(1,t) = 0, \\ &v_y(0,t) + \beta\gamma(t)\,|v_t(0,t)|^\rho\, v_t(0,t) = 0, \qquad t > 0. \end{aligned} \tag{2.7}$$

Here,

$$\begin{aligned} a_1(y,t) &= \frac{k^2 - [y\gamma'(t) + \alpha_1'(t)]^2}{\gamma^2(t)}, \\ a_2(y,t) &= \frac{y\gamma'(t) + \alpha_1'(t)}{\gamma(t)}, \\ b_1(y,t) &= \frac{\partial a_2}{\partial t}, \qquad f_1(y,t) = f(x(y,t),t). \end{aligned}$$

Remark 1 *It is easy to verify that under the conditions of Theorem 1 is diffeomorfism and preserves hyperbolicity of the problem* (2.5) − (2.7)*. It means that solvability of* (2.5) − (2.7) *implies solvability of* (2.1) − (2.3)*.*

Theorem 2 *Let all the conditions of Theorem 1 hold, then there exists a unique strong solution of* (2.5) − (2.7) *from the class:*

$$\begin{aligned} &v \in L^\infty(0,T;V(0,1) \cap H^2(0,1)), \\ &v_t \in L^\infty(0,T;V(0,1)), \\ &v_{tt} \in L^\infty(0,T;L^2(0,1)). \end{aligned}$$

To prove the existence of Theorem 2, we will use the method of Galerkin.

3. The Method of Galerkin

In order to exploit the Galerkin method, we introduce a new unknown function z such that $z(y,0) = z_t(y,0) = 0$. In fact, let $\psi = v_0(y) + tv_1(y)$, then $z = v - \psi$ satisfies the following problem:

$$z_{tt} - a_1 z_{yy} - 2a_2 z_{yt} - b_1 z_y = f_2(y,t), \qquad \text{in} \quad Q, \tag{3.1}$$

$$z(y,0) = z_t(y,0) = 0, \tag{3.2}$$

$$z(1,t) = 0, \tag{3.3}$$

$$z_y(0,t) + \beta\gamma(t)\,|z_t(0,t) + v_1(0)|^\rho\,(z_t(0,t) + v_1(0)) + v_0'(0) + tv_1'(0) = 0, \tag{3.4}$$

where $f_2(y,t) \equiv f_1(y,t) + a_1\psi_{yy} + 2a_2\psi_{yt} + b_1\psi_y$.

Let $\{w_j(y)\}$ be a basis in $V(0,1)$ orthonormal in $L^2(0,1)$. We define approximate solutions to (2.1) − (2.4) in the form:

$$z^N(y,t) = \sum_{j=1}^{N} g_j^N(t) w_j(y).$$

The unknown functions $g_j^N(t)$ will be find as solutions to the following Cauchy Problem:

$$\left(z_{tt}^N, w_j\right)(t) + \left(a_1 z_y^N, w_{jy}\right)(t) + \left(a_{1y} z_y^N, w_j\right)(t) - 2\left(a_2 z_{yt}^N, w_j\right)(t)$$

$$+a_1\left\{\beta\gamma(t)\left|z_t^N(0,t)+v_1(0)\right|^\rho\left(z_t^N(0,t)+v_1(0)\right)+v_0'+tv_1'(0)\right\}w_j(0)$$

$$-\left(b_1 z_y^N, w_j\right)(t) = \left(f_2, w_j\right)(t), \tag{3.5}$$

$$g_j^N(0) = g_{jt}^N(0) = 0, \quad j = 1, 2, \ldots, N. \tag{3.6}$$

The system (3.5) is a normal system of N ordinary differential equations, therefore, by the Caratheodery's Theorem, (3.5) – (3.6) has solutions at some interval $(0, T_N)$. It means that the approximations $z^N(y,t)$ exist at the same interval. To prolong them at the whole interval $(0,T)$ and to pass to the limit in (3.5) as $N \to \infty$, we need a priori estimates of z^N.

3.1. First Estimate

Substituting in (2.5) w_j by z_t^N, after some calculations we come, omitting the index N, to the identity:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 \left(z_t^2(y,t) + a_1 z_y^2(y,t)\right)dy + \int_0^1 \left(a_{1y} - b_1\right) z_y z_t dy$$

$$-\left[\frac{k^2 - \alpha_1'^2(t)}{\gamma^2(t)}\right]\left[\beta\gamma(t)\left|z_t + v_1\right|^\rho\left(z_t + v_1\right) + v_0' + tv_1'\right] z_t\Big|_{y=0}$$

$$\int_0^1 \left(a_{2y} z_t^2 - \frac{a_{1t}}{2} z_y^2\right)dy + \frac{\alpha_1'(t)}{\gamma(t)} z_t^2(0,t) = \int_0^1 f_2 z_t dy.$$

Using the assumptions of Theorem 1 and the Young's inequality, we reduce it to the inequality:

$$\frac{\partial E_1(t)}{\partial t} + |\beta| K_0 \gamma(t)\left|z_t(0,t) + v_1(0)\right|^{\rho+2} \le C_1 E_1(t) + C_2 + \int_0^1 f_2^2 dy, \tag{3.7}$$

where $E_1(t) = \int_0^1 \left(z_t^2 + a_1 z_y^2\right)dy$, $K_0(t) = \left[\frac{k^2 - \alpha_1'^2(t)}{\gamma^2(t)}\right]$; positive constants C_1, C_2 are defined by v_0, v_1 and coefficients a_1, a_2, b_1.

By Gronwall's lemma:

$$E_1(t) \le C, \quad \forall t \in (0,T). \tag{3.8}$$

Returning to (3.7), we get:

$$|\beta| \int_0^t K_0(\tau)\gamma(\tau)\left|z_\tau(0,\tau) + v_1(0)\right|^{\rho+2} d\tau \le C \tag{3.9}$$

and the constants C_1, C_2, C do not depend on N, $t \in (0,T)$.

3.2. Second Estimate

Differentiating (3.5) with respect to t, substituting in the result w_j by z_{tt}^N, we come to the equality:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1 \left[z_{tt}^2 + a_1 z_{yt}^2\right] dy + \frac{\partial}{\partial t}\int_0^1 \left[a_{1t} z_y z_{yt} + 2a_{2t} z_t z_{yt}\right] dy$$

$$+\int_0^1 \left[a_{1yt} z_y z_{yt} - 2a_{2t} z_{yt} z_{tt} + 2a_{2yt} z_t z_{tt} + \frac{3}{2} a_{2y} z_{tt}^2 - b_{1t} z_y z_{tt} - b_1 z_{yt} z_{tt}\right] dy$$

$$-\left[h_1'(t)\left|z_t + v_1\right|^\rho (z_t + v_1) + h_1(t)(\rho+1)\left|z_t + v_1\right|^\rho z_{tt} + h_2'(t)\right] z_{tt}\Big|_{y=0}$$

$$+\left[h_3'(t) z_t(0,t) + h_3(t) z_{tt}(0,t)\right] z_{tt}(0,t) = \int_0^1 f_{2t} z_{tt} dy, \tag{3.10}$$

where:
$h_1(t) = \beta K_0(t)\gamma(t)$,
$h_2(t) = K_0(t)\psi_y(0,t)$,
$h_3(t) = 2\dfrac{\alpha_1'(t)}{\gamma(t)} z_t(0,t) z_{tt}(0,t)$.

Putting in (3.5) $t = 0$, substituting w_j by $z_{tt}^N(0,t)$ and taking into account assumption 2 of Theorem 1, we obtain:

$$\int_0^1 \left|z_{tt}(y,0)\right|^2 dy \le C, \tag{3.11}$$

where C does not depend on N.

Integrating (3.10) over $(0,T)$, taking into account the estimates (3.8), (3.9), (3.11), we get:

$$E_2(t) \equiv \int_0^1 \left(z_{tt}^2 + a_1 z_{yt}^2\right) dy \le C\left(t + \int_0^t E_2(\tau) d\tau\right).$$

The Gronwall's lemma implies:

$$E_2(t) \le C, \quad \forall t \in (0,T), \tag{3.12}$$

where the constant C does not depend on N. Returning to (3.10), we get:

$$\int_0^\tau \left|z_t^N(0,t) + v_1(0)\right|^\rho \left(z_{tt}^N(0,t)\right)^2 dt \le C, \quad \forall \tau \in (0,T). \tag{3.13}$$

4. Existence of Strong Solutions

It follows from the estimates (3.8), (3.12) that:

$$\begin{aligned} z^N &\longrightarrow z \quad \text{weak star in } L^\infty(0,T;V(0,1)), \\ z_t^N &\longrightarrow z_t \quad \text{weak star in } L^\infty(0,T;V(0,1)), \\ z_{tt}^N &\longrightarrow z_{tt} \ \text{weak star in } L^\infty(0,T;L^2(0,1)). \end{aligned} \tag{4.1}$$

Moreover, (3.3) and (3.13) imply that:

$$\left(z_t^N(0,t)+v_1(0)\right)^{\frac{\rho}{2}+1} \in H^1(0,1) \subset C[0,T],$$

therefore:

$$\left|z_t^N(0,t)+v_1(0)\right|^{\frac{\rho}{2}+1} \underset{C(0,T)}{\Rightarrow} \left|z_t(0,t)+v_1(0)\right|^{\frac{\rho}{2}+1}. \tag{4.2}$$

The limits (4.1) and (4.2) permit us to pass to the limit as $N \to \infty$ in (3.5) and obtain for a. l. $t \in (0,T)$:

$$\begin{aligned} &(z_{tt},\phi)(t) + (a_1 z_y, \phi_y)(t) + (a_{1y} z_y, \phi)(t) - 2(a_2 z_{yt}, \phi)(t) - (b_1 z_y, \phi)(t) \\ &- \left[\frac{k^2 - \alpha_1'^2(t)}{\gamma^2(t)}\right] \{\beta\gamma(t)\left|z_t(0,t)+v_1(0)\right|^\rho (z_t(0,t)+v_1(0)) + v_0' + tv_1'(0)\}\,\phi(0) \\ &= (f_2,\phi)(t), \end{aligned} \tag{4.3}$$

where $\phi(y)$ is an arbitrary function from $V(0,1)$. It means that $z(y,t)$ is a generalized solution (3.1) – (3.4) and we may rewrite (4.3) for a. l. $t \in (0,T)$ in the form:

$$(z_y, (a_1\phi)_y)(t) + G(t)\phi(0) = (F,\phi)(t), \tag{4.4}$$

where $G(t)$ and $F(y,t)$ are known functions and $f \in L^\infty(0,T;L^2(0,1))$. Because $z(1,t)=0$ and $z_y(0,t) = \dfrac{G(t)}{a_1(0,t)}$, it follows from (4.4) that $z(y,t)$ is a generalized solution to the following boundary value problem for the ordinary differential equation:

$$-z_{yy} = \frac{F(y,t)}{a_1(y,t)},$$

$$z(1,t)=0, \qquad z_y(0,t) = \frac{G(t)}{a_1(0,t)},$$

hence $z_{yy} \in L^\infty(0,T;L^2(0,1))$. This proves the existence part of Theorem 2 ■

5. Uniqueness of the Solution

Suppose that there exist two solutions $z_1(y,t)$ and $z_2(y,t)$ of the problem (2.4) – (2.6).

Putting $z(y,t) = z_1(y,t) - z_2(y,t)$, we obtain:

$$z_{tt} - a_1 z_{yy} - 2a_2 z_{yt} - b_1 z_y = f_1(y,t),$$

$$z(y,0) = 0,$$
$$z_t(y,0) = 0,$$

$$z(1,t) = 0,$$
$$z_y(0,t) + \beta\gamma(t)\Big[|z_{1t}(0,t)|^\rho z_{1t}(0,t) - |z_{2t}(0,t)|^\rho z_{2t}(0,t)\Big] = 0.$$

Proceeding as by proving the First estimate and observing that the function $|v_t|^\rho\, v_t$ is monotonous, we obtain the inequality:

$$E_1(t) \le C\int_0^t E_1(\tau)d\tau,$$

where $E_1(t) = \int_0^1 \left(z_t^2(y,t) + a_1 z_y^2(y,t)\right) dy$.

By the Gronwall's inequality:

$$E_1(t) \equiv 0.$$

This implies that $z_1(y,t) = z_2(y,t)$ and, therefore, the solution of the problem (2.4) – (2.6) is unique. This complete the proof of Theorem 2 ■

Remark 2 *Assertions of Theorem 1 follow directly from Theorem 2 and Remark 1.*

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