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On a Combinatorial Result Related to the Rogers-Ramanujan Identities

J.P.O. SANTOS[1], Departamento de Matemática Aplicada, IMECC-UNICAMP, Cx.P. 6065, 13081-970, Campinas, SP, Brasil

P. MONDEK[2], Departamento de Matemática, CCET-UFMS, Cx.P. 549, 79070-900, Campo Grande, MS, Brasil.

Abstract. We give a generating function for partitions with difference conditions and a combinatorial proof for a bijection between these partitions and another class of partitions. New combinatorial interpretations for the Rogers-Ramanujan identities are included as special cases.

1. Introduction

We begin presenting some basic concepts: A partition of a positive integer n is a finite nonincreasing sequence of positive integers $m_1,\ldots,m_r$ such that $m_1 + \ldots + m_r = n$.

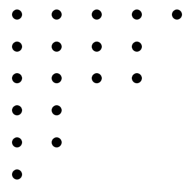
Generating functions are used for studying partitions. For many problems it suffices to consider these functions as "formal power series." For others one requires that they be analytic functions of complex variables. For instance, if we denote $p(n)$ as the number of partitions of n for each n, then the generating function for $p(n)$ is given by the following analytic identity:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1}, \tag{1.1}$$

where $|q| < 1$.

Another very useful device for studying partitions is the graphic representation of the partition of an integer n. The Ferrers graph of a partition is a graphical representation which associates each summand m of a partition with a row of m dots. Thus, the Ferrers graph of the partition $5 + 4 + 4 + 2 + 2 + 1$ of 18 is

[1]josepli@ime.unicamp.br
[2]mondek@hilbert.dmt.ufms.br



In [1], page 59, Andrews presents a bijective proof, given by Bressoud, for the following theorem:

Theorem A *The number of partitions of n with minimal difference at least 2 between parts equals the number of partitions of n into distinct parts wherein each even part is larger than twice the number of odd parts.*

The Rogers-Ramanujan identities are:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}, \tag{1.3}$$

where we are using the standard notation

$$\begin{aligned} (a;q)_0 &= 1 \\ (a)_n = (a;q)_n &= (1-a)(1-aq)\dots(1-aq^{n-1}), \ n > 0. \end{aligned}$$

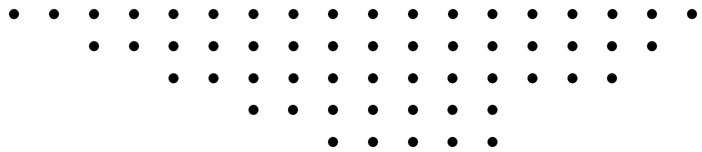
It is clear that Theorem A is related to the first Rogers-Ramanujan identity since the left side of (1.2) is the generating function for partitions as described in the first part of Theorem A.

The general result that we are going to prove has as special case, not only this Theorem A, but also one related to the second Rogers-Ramanujan identity which is the following:

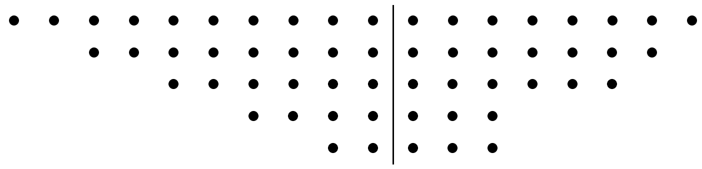
Theorem 1 *The number of partitions of n with minimal difference at least 2 between parts, with parts greater than 1 equals the number of partitions of n into distinct parts wherein each odd part is larger than 2 plus twice the number of even parts.*

The proof for this theorem is similar to the one given by Bressoud [3] for Theorem A.

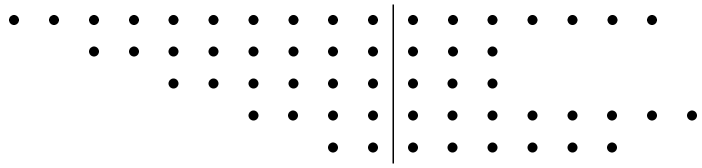
Proof. We consider a partition π as described in the first part of the theorem. We represent π with a modified Ferrers graph in which we indent each row by two nodes. Thus if $\pi : 18 + 15 + 12 + 7 + 5$, our representation is:



We now put a vertical bar in our graph so that to the left are rows of 2, 4, 6, 8, etc nodes going from botton to top.



We reorder the rows to the right of the bar putting first the rows with an odd number of nodes (in descending order) and then the rows with an even number of nodes (in descending order). Thus our new graph is:



and reading the new complete rows as parts of a transformed partition we have in this instance $17+12+11+9+8$.

It is immediate from our construction that all parts are distinct and that the smallest odd part is larger than 2 plus twice the number of even parts. The process is clearly reversible thus giving us a bijection between the two classes of partitions presented in the Theorem. ∎

2. The Main Result

We state, next, our main theorem.

Theorem 2 *Let $A(n,\ell)$ be the number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_s$ such that $b_j - b_{j+1} \geq 2$ and $b_s > \ell$, and $B(n,\ell)$ be the number of partitions of n in distinct parts such that the smallest part is greater than ℓ and each part$\equiv \ell (mod\ 2)$ is greater than $2t+\ell+1$ where t is the number of parts $\equiv \ell + 1 (mod\ 2)$. Then, for $\ell \geq 0$, $A(n,\ell) = B(n,\ell)$ for all n and*

$$\sum_{n=0}^{\infty} A(n,\ell) q^n = \sum_{s=0}^{\infty} \frac{q^{s^2+\ell s}}{(q)_s} .$$

Proof. Let $n = b_1 + b_2 + \cdots + b_s$ be a partition enumerated by $A(n,\ell)$. If we substract $\ell+1$ from $b_s, \ell+3$ from $b_{s-1}, \ldots, \ell+(2s-1)$ from b_1 we are left with a partition of $n-(\ell+1+\ell+3+\cdots+\ell+(2s-1)) = n-\ell s-s^2$ in at most s parts and this is generated by

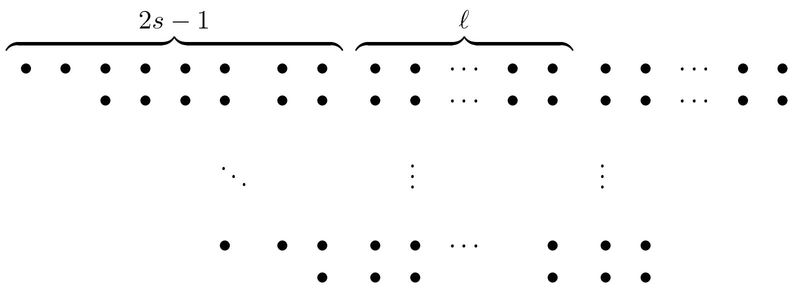
$$\frac{q^{s^2+\ell s}}{(q)_s}, s \geq 1.$$

Hence the generating function for the partitions enumerated by $A(n,\ell)$ is

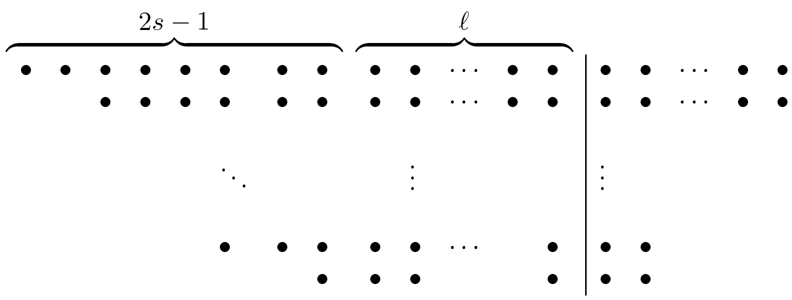
$$1+\sum_{s=1}^{\infty}\frac{q^{s^2+\ell s}}{(q)_s}.$$

Now in order to prove that $A(n,\ell) = B(n,\ell)$ we are going to construct a bijection between the sets enumerated by these two numbers.

We take a partition π enumerated by $A(n,\ell)$. Considering that the difference between parts is at least 2 we may represent π with a modified Ferrers graph in which we indent each row by two nodes and, in doing so, our representation is:



We now put a vertical bar in our graph so that to the left are rows of $\ell+1, \ell+3, \ldots, \ell+(2s-1)$ nodes going from botton to top.



Now we reorder the rows to the right of the bar putting first the rows with an odd number of nodes and after the rows with an even number of nodes, both in descending order. If we consider now the new rows as parts of a transformed partition it is easy to see that from our construction all parts are distinct, each one is greater than ℓ and the smallest

part$\equiv \ell(mod\ 2)$ is greater than $2t+\ell+1$ where t is the number of parts $\equiv \ell+1(mod\ 2)$. In fact if there are r parts$\equiv \ell(mod\ 2)$ then the r-th is $\geq 2(s-r)+\ell+2 > 2(s-r)+\ell+1$.

What we have described is clearly reversible thus giving us a bijection between the two classes of partitions enumerated by $A(n,\ell)$ and $B(n,\ell)$. ■

We illustrate, below, the partitions enumerated by $A(n,\ell)$ and $B(n,\ell)$ and the correspondence between them given by the bijection described in the theorem for $n=19$ and $\ell=2$.

$A(19,2)$		$B(19,2)$
19	$\longleftrightarrow$	19
$16+3$	$\longleftrightarrow$	$16+3$
$15+4$	$\longleftrightarrow$	$13+6$
$14+5$	$\longleftrightarrow$	$14+5$
$13+6$	$\longleftrightarrow$	$11+8$
$12+7$	$\longleftrightarrow$	$12+7$
$11+8$	$\longleftrightarrow$	$10+9$
$11+5+3$	$\longleftrightarrow$	$11+5+3$
$10+6+3$	$\longleftrightarrow$	$10+6+3$
$9+7+3$	$\longleftrightarrow$	$9+7+3$
$9+6+4$	$\longleftrightarrow$	$8+6+5$

The cases $\ell=0$ and $\ell=1$ are the special cases described in Theorem A and Theorem 1, respectively, that are related to the Rogers-Ramanujan identities.

We observe that if in the proof of Theorem 2 we reorder putting first the even ones we get the following result:

Theorem 3 *Let $C(n,\ell)$ be the number of partitions of n in distinct parts greater than ℓ such that each part $\equiv \ell+1(mod\ 2)$ is greater than $2r+\ell$ where r is the number of parts $\equiv \ell(mod\ 2)$. Then $C(n,\ell)=A(n,\ell)$ for $\ell\geq 0$.*

Comment: Except for the cases $\ell=0$ and $\ell=1$ any interesting infinite product representation for the given generating functions in Theorem 2 is not known.

References

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