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Exact Barrier Option Valuation with Deterministic Volatility†

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ABSTRACT. Focus, in the past four decades, has been obtaining closed-form expressions for the no-arbitrage prices and hedges of modified versions of the European options, allowing the dynamic of the underlying assets to have non-constant parameters. In this paper, we obtain a closed-form expression for the price and hedge of an up-and-out European barrier option, assuming that the volatility in the dynamic of the risky asset is an arbitrary deterministic function of time. Setting a constant volatility, the formulas recover the Black and Scholes results, which suggests minimum computational effort. We introduce a novel concept of relative standard deviation for measuring the exposure of the practitioner to risk (enforced by a strategy). The notion that is found in the literature is different and looses the correct physical interpretation. The measure serves aiding the practitioner to adjust the number of rebalances during the option's lifetime.

Keywords: barrier option, no-arbitrage pricing, hedging, Martingale measure, time-change for Martingales.

1 INTRODUCTION

Under arbitrage-free assumptions, Black & Scholes [1] and Merton [17] pioneered the achievements on pricing and hedging derivatives in financial markets. They considered an European call option and a market with one bond and one stock where the parameters in the dynamics that model the market have constant values. By its turn, Harrison & Pliska [4] – among others – showed that, essentially, there is an equivalence between absence of arbitrage opportunities and the existence of an equivalent measure that renders the discounted underlying stock a martingale: under this measure, pricing a derivative is allowed to be naively obtained, in that average is applied to the discounted payoff, conditional to the present information.

Underpinned by these seminal results, significative advances followed in obtaining closed-form expressions for the exact prices and hedges of options, as can be verified, for instance, in the

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works of Heston [5] addressing the Ornstein-Uhlembeck format for stochastic volatility, Cox & Ross [2] focusing CEV (constant elasticity of variance) models, and Lo et al. [15] for CEV with deterministic time-dependent coefficients. The case of pure deterministic time-dependent coefficients are direct extensions of [1] (see, e.g., [12]). But the results and techniques do not replicate if entering with more sophisticated derivatives as path dependent options, particularly barrier options. Indeed, achievements in terms of closed-form pricing formulas are more restricted in this case.

Barrier options are standard European options which involve a barrier constraint tailored for a certain behavior of the underlying asset along the option's lifetime. If this behavior is what the investor thinks will happen, then he may pay less buying the barrier option instead of its standard counterpart (the European option), obtaining the same result whenever his beliefs meet reality. Otherwise the option's payoff cancels (see, e.g., [3, 12]). Barrier options have been present for more than two decades in the foreign exchange, equity and commodity markets and became very popular in recent years.

Merton [17] derived a closed-form solution for the price of a down-and-out European call option with constant barrier. Explicit solutions for arbitrage-free prices of some other types of barrier options are found in the work of Rubinstein & Reiner [20]. Rich [18] derived closed-form solutions for European barrier options with a fixed rebate and constant barrier, as well as exponential barriers with constant time coefficient on the exponent. Heynen & Kat [6, 7] derived closed-form pricing formulas for partial barrier options, and analytic valuation formulas for outside barrier options as well. Via a modified version of the method of images, Kwok et al. [11] extended this scope allowing more than one underlying asset and barriers that can be constant or exponential with a constant time coefficient. Kolkiewicz [9] addressed several types of double-barrier options where an infinite series representation for the price is given. Relying on the hitting probabilities of a Brownian motion technique, closed-form expressions for European style double barrier options are also encountered in [16]. Using probabilistic techniques, Kunitomo & Ikeda [10] evaluated options contracts monitored by two knock-out barriers with exponential format, whose valuation formulas are given in terms of an infinite series representation. The works above assume that the market model has constant parameters. Allowing the parameters in the dynamics to be deterministic functions of time, Roberts & Shortland [19] explicitly derived approximation formulas for the prices of barrier options by means of estimating the boundary crossing times of the asset price and the technique of hazard rate tangent approximations. Lo et al. [13], via the method of images, obtained estimates of prices of barrier options with generic (square integrable) trajectories. The estimates stem from a parameterized class of trajectories which spontaneously arises from the technique and for which the prices are exact. Considering the CEV models with time-dependent coefficients, Lo et al. [14] also obtained estimates for the prices of barrier options with generic trajectories, brought out by a parameterized class of trajectories for which the prices are exact.

In this note, we present closed-form expressions for the exact price and hedging strategy of an up-and-out call option with a constant barrier B. The barrier tests whether the price $S(u)$ of the risky asset – which evolves according to the stochastic differential equation

$$dS(u) = \mu(u)S(u)du + \sigma(u)S(u)dW(u) \tag{1.1}$$

– agrees with $S(u) \leq B \ \forall u \in [0, T]$. In the above expression the mean rate of return (drift) μ and the volatility σ are arbitrary deterministic functions of time subject to the mild constraint of square integrability (jumps are allowed, for instance). W is a standard Brownian Motion and T is the expiration time. We assume the riskless interest rate $r = 0$.

We justify modeling the market via a deterministic time dependent volatility noticing the following. Market's forecasts stem primarily from practitioners' believes and intuition, which in turn lead to deterministic scenarios. Substantiated by these scenarios, practitioners become more confident to choose strategies and formulate derivatives. Whence, modeling the market via deterministic time dependent parameters is a worthy guiding reference.

It is noteworthy that the version of the problem treated herein with a non-zero riskless interest rate *does not* allow closed-form expressions. We also note that the prices herein serve as a worst case benchmark to [14] set to a constant value barrier; this is so since the function that spans the parameterized class of exactly priced barriers there approaches zero as the CEV model with time dependent coefficient tends to the pure time dependent model.

To the best of the authors knowledge, the work in this note is not present in the literature; in particular, the constant value barrier is neither in the core of the exactly priced barriers options of [14] (for which our dynamics is a limiting case), nor that of [13].

In terms of simulations, we consider hedging a short position via a strategy that creates a long position in the option synthetically by buying or selling shares of the asset a number of times per day. We introduce a novel concept of relative standard deviation for measuring correctly the exposure of the practitioner to risk (enforced by the strategy), or else, the efficiency of risk absorbtion assigned to the strategy. The notion that is present in the literature and that is exploited in the financial industry is different and looses the correct physical interpretation. The measure serves aiding the practitioner to adjust the number of rebalances during the option's lifetime.

2 PRICING AND HEDGING RESULTS

We consider a probability space $(\Omega, \mathcal{F}, \tilde{\mathbf{P}})$ where $\tilde{\mathbf{P}}$ is the risk-neutral probability for the market described in Section 1. Under $\tilde{\mathbf{P}}$, the price of the underlying risky asset evolves according to the stochastic differential equation

$$dS(u) = \sigma(u)S(u)d\tilde{W}(u) \tag{2.1}$$

and the initial price value $S(0)$. $\tilde{W}(u)$, $0 \leq u \leq T$, is a standard Brownian Motion and the volatility $\sigma(u)$ is an arbitrary square integrable deterministic function of time. The barrier's payoff we are interested in reads

$$H(T) = (S(T) - K)^{+}\mathbf{I}_{\{S(u)\leq B \ \forall \ u\in[0,T]\}}, \tag{2.2}$$

where K is the strike price and $0 < K < B$.

At time $t \in [0, T]$ arbitrarily fixed, the no-arbitrage price for this option is (see, e.g., [4], [21])

$$H(t) = \tilde{E}\left[H(T) \mid \mathcal{F}(t)\right], \tag{2.3}$$

with $\mathcal{F}(t) \subset \mathcal{F}$, $0 \leq t \leq T$, denoting the filtration generated by W.

In order to deal with a zero-origin starting point, define $\sigma_t(u) = \sigma(t+u)$, $S_t(u) = S(t+u)$ and the standard Brownian Motion $\tilde{W}_t(u) = \tilde{W}(t+u) - \tilde{W}(t)$, $0 \leq u \leq T-t$. Thus, from (2.1),

$$dS_t(u) = \sigma_t(u) S_t(u) d\tilde{W}_t(u), \tag{2.4}$$

where the initial price value is $S_t(0) = S(t)$. So, the risky asset price process $S_t(u)$ is a martingale which reads $S_t(u) = S(t)\exp\{\hat{I}_t(u)\}$, where

$$\hat{I}_t(u) = \int_0^u \sigma_t(s) d\tilde{W}_t(s) - \frac{1}{2}\int_0^u \sigma_t^2(s)ds, \;\; 0 \leq u \leq T-t. \tag{2.5}$$

Also define

$$\hat{M}_t = \max_{0 \leq u \leq T-t} \hat{I}_t(u), \quad \text{so that} \quad \max_{0 \leq u \leq T-t} S_t(u) = S(t)\exp\{\hat{M}_t\}. \tag{2.6}$$

Moreover, (2.6) allows us to write

$$\begin{aligned} \{S(u) \leq B \;\; \forall u \in [t,T]\} &= \{S_t(u) \leq B \;\; \forall u \in [0, T-t]\} \\ = \left\{ \max_{0 \leq u \leq T-t} S_t(u) \leq B \right\} &= \left\{ S(t)\exp\left\{\hat{M}_t\right\} \leq B \right\}, \end{aligned}$$

so the payoff $H(T)$ reads

$$H(T) = \Big(S(t)\exp\{\hat{I}_t(T-t)\} - K\Big) \mathbf{I}_{\left\{\hat{I}_t(T-t) \geq k(t),\; \hat{M}_t \leq b(t)\right\}}, \tag{2.7}$$

with $k(t) = \ln\left(\frac{K}{S(t)}\right)$ and $b(t) = \ln\left(\frac{B}{S(t)}\right)$.

Lemma 2.1 gives the joint density function for calculating (2.3).

Lemma 2.1. *Under $\tilde{\mathbf{P}}$, the joint density function of the pair of random variables $\left(\hat{M}_t, \hat{I}_t(T-t)\right)$ is*

$$\tilde{f}_{\hat{M}_t, \hat{I}_t(T-t)}(x,y) = \begin{cases} \dfrac{2(2x-y)}{h(t)\sqrt{2\pi h(t)}} \exp\left\{-\dfrac{1}{2}y - \dfrac{1}{8}h(t) - \dfrac{(2x-y)^2}{2h(t)}\right\}, & y \leq x, \;\; x > 0 \\ 0, & \textit{otherwise} \end{cases}, \tag{2.8}$$

where $h(t) = \int_t^T \sigma^2(u)du$.

Proof. Underpinned by Girsanov's Theorem, we obtain the joint density function for $\big(\hat{M}_t, \hat{I}_t(T-t)\big)$ under an auxiliary measure $\hat{\mathbf{P}}$ which renders the process $\hat{I}_t(u)$ given by (2.5) a continuous martingale, namely,

$$\hat{I}_t(u) = \int_0^u \sigma_t(s) d\hat{W}(s), \;\; 0 \leq u \leq T-t, \tag{2.9}$$

where

$$\hat{W}(u) = \tilde{W}_t(u) - \frac{1}{2}\int_0^u \sigma_t(s)ds, \tag{2.10}$$

is a Brownian Motion under $\hat{\mathbf{P}}$. Indeed, defining

$$Z_t(u) = \exp\left\{\int_0^u \frac{\sigma_t(s)}{2} d\tilde{W}_t(s) - \frac{1}{2}\int_0^u \frac{\sigma_t^2(s)}{4} ds\right\}, \ 0 \le u \le T-t, \tag{2.11}$$

and relying on the square integrability of $\sigma(u)$, we have that $Z_t(u)$ is a Radon-Nikodým derivative process and

$$\hat{\mathbf{P}}(A) = \int_A Z_t(T-t) d\tilde{\mathbf{P}}, \quad A \in \mathcal{F} \tag{2.12}$$

defines a new measure under which $\hat{W}$ is a standard Brownian Motion, so that $\hat{I}_t$ is a continuous martingale under $\hat{\mathbf{P}}$. Now, relying on the Time-Change for Martingales Theorem (see, e.g. [8]), we may further express $\hat{I}_t$ as

$$\hat{I}_t(u) = \mathcal{W}(h_t(u)), \ 0 \le u \le T-t, \tag{2.13}$$

where $\mathcal{W}$ is a Brownian Motion under $\hat{\mathbf{P}}$, and

$$h_t(u) = \left[\hat{I}_t, \hat{I}_t\right](u) = \int_0^u \sigma_t^2(s) ds \tag{2.14}$$

is the quadratic variation of $\hat{I}_t(u)$. Moreover, since $h_t(u)$ is continuous and increasing, it follows from (2.6) that

$$\hat{M}_t = \max_{0 \le s \le h_t(T-t)} \mathcal{W}(s). \tag{2.15}$$

Since $\sigma_t(u)$ is deterministic, and relying on (2.13), (2.15) and on the Reflection Principle for Brownian Motion, we have, for $y \le x$ and $x > 0$, that

$$\hat{\mathbf{P}}\left\{\hat{M}_t \ge x, \hat{I}_t(T-t) \le y\right\} = \hat{\mathbf{P}}\left\{\hat{I}_t(T-t) \ge 2x - y\right\}, \quad y \le x, \quad x > 0, \tag{2.16}$$

which stands for a version of the Reflection Principle for Brownian Motion extended for Continuous-time Martingales with Deterministic Quadratic Variation – in this case given by (2.14). Differentiating (2.16) with respect to x and y, we obtain the joint density function of the pair $\left(\hat{M}_t, \hat{I}_t(T-t)\right)$ under the auxiliary measure $\hat{\mathbf{P}}$, given by

$$\hat{f}_{\hat{M}_t, \hat{I}_t(T-t)}(x, y) = \frac{2(2x-y)}{h_t(T-t)\sqrt{2\pi h_t(T-t)}} \exp\left\{\frac{-(2x-y)^2}{2h_t(T-t)}\right\}, \ y \le x, \ x > 0, \tag{2.17}$$

where we used the fact that $\hat{I}_t(T-t)$ is normally distributed with zero mean and variance $h_t(T-t)$. By its turn, (2.5), (2.10), (2.11) and (2.14) gives us that

$$Z_t(T-t) \quad = \quad \exp\left\{\frac{1}{2}\hat{I}_t(T-t) + \frac{1}{8}h_t(T-t)\right\},$$

so that

$$\begin{aligned} \tilde{\mathbf{P}}\left\{\hat{M}_t \le x, \hat{I}_t(T-t) \le y\right\} &= \hat{E}\left[\frac{1}{Z_t(T-t)} \mathbf{I}_{\{\hat{M}_t \le x, \hat{I}_t(T-t) \le y\}}\right] \\ &= \int_{-\infty}^{y}\int_{-\infty}^{x} \exp\left\{-\frac{1}{2}v - \frac{1}{8}h_t(T-t)\right\} \hat{f}_{\hat{M}_t, \hat{I}_t(T-t)}(w, v)\, dw\, dv. \end{aligned} \tag{2.18}$$

Noticing that $h(t) = h_t(T-t)$ and reminding (2.17), the result follows differentiating (2.18) with respect to y and x. $\square$

The pricing and hedging result is the content of the following theorem.

Theorem 2.1. *Consider an up-and-out call option with strike price K, time of expiration $T > 0$ and a constant barrier B such that $0 < K < B$. Also assume that the volatility $\sigma(t)$ that enters in the dynamic of the risky asset is an arbitrary square integrable deterministic function of time and the interest rate of the riskless asset is zero. If the option has not knocked out prior to time $t \in [0, T)$, then the no-arbitrage price $H(t)$ and hedge $\Delta(t)$ of this option are given by*

$$\begin{aligned} H(t) &= S(t)[N(z^+_{t,a}) - N(z^+_{t,b})] - K[N(z^-_{t,a}) - N(z^-_{t,b})] \\ &\quad - B[N(z^+_{t,c}) - N(z^+_{t,d})] + K\,b \cdot [N(z^-_{t,c}) - N(z^-_{t,d})], \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \Delta(t) &= [N(z^+_{t,a}) - N(z^+_{t,b})] + q[N(z^-_{t,c}) - N(z^-_{t,d})] + h(t)^{-1/2}\{[-N'(z^+_{t,b}) \\ &\quad + a^{-1}N'(z^-_{t,b})] + d[N'(z^+_{t,c}) - N'(z^+_{t,d})] - q[N'(z^-_{t,c}) - N'(z^-_{t,d})]\}. \end{aligned} \tag{2.20}$$

In the above equations, $S(t)$ is the price of the risky asset observed at time t, $a = S(t)/K$, $b = S(t)/B$, $c = B^2/KS(t)$, $d = B/S(t)$, $q = K/B$,

$$h(t) = \int_t^T \sigma^2(u)du, \qquad z^{\pm}_{t,s} = \frac{1}{\sqrt{h(t)}}\left[\ln s \pm \frac{1}{2}h(t)\right],$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2}}dy \quad \textit{and} \quad N'(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}.$$

Proof. Bearing in mind that $S(t)$ is $\mathcal{F}(t)$-measurable and $\hat{I}_t(T-t)$ and $\hat{M}_t$ are independent of $\mathcal{F}(t)$, it follows, from (2.3) and (2.7), that

$$\begin{aligned} H(t) &= \tilde{E}\left[\left(S(t)\exp\{\hat{I}_t(T-t)\} - K\right)\mathbf{I}_{\left\{\hat{I}_t(T-t)\ge k(t),\ \hat{M}_t \le b(t)\right\}} \mid \mathcal{F}(t)\right] \\ &= \int_{k(t)}^{b(t)}\int_{y^+}^{b(t)} \left(S(t)e^y - K\right)\tilde{f}_{\hat{M}_t,\hat{I}_t(T-t)}(x,y)\,dx\,dy, \end{aligned} \tag{2.21}$$

where $\tilde{f}_{\hat{M}_t,\hat{I}_t(T-t)}$ is given by (2.8) and $y^+ = \max\{y, 0\}$. The pricing formula (2.19) follows after some algebraic manipulation, while the derivative of this price with respect to the risky asset price leads us to the hedging strategy formula. $\square$

Setting the volatility a constant recovers verbatim the classical barrier pricing and hedging formulas with $r = 0$; letting, in addition, the barrier go to infinity recovers the Black and Scholes classical result for an European call option.

3 SIMULATIONS

The numerical results in this section were generated via the Monte Carlo Method in conjunction with the the Antithethic Variates Method. One million simulations were performed. We consider hedging a short position in an up-and-out call option with $T = 20$ days, $S(0) = 100$, $K = 96$, and a barrier $B = 110$. The term structure of the volatility that we arbitrarily chose exhibits a jump at time $T/2$. More explicitly,

$$\sigma_1(t) = \begin{cases} 0.5 \quad , & 0 \le t \le T/2 \\ 0.24 - t \quad , & T/2 < t \le T. \end{cases} \tag{3.1}$$

For the sake of simplicity we set a constant value for μ, namely, $\mu = 10\%$.

With the aim of hedging the short position, we consider a strategy that creates a long position in the option synthetically by buying or selling a certain quantity of shares given by the difference between the actual and the previous delta values (which stems from the theoretical calculations). Table 1 illustrates one realization of this delta hedging scheme considering a contract of one option. The hedging scheme performs three times per day, with the option being knocked out on the first rebalance (adjustment) of the fourth day. The asset price evolves as in column 1. Column 3 stems from column 2 and shows the buying or selling strategy. The cost of shares purchased (no transactions costs were assumed), as in column 4, creates a debt/credit in a bank account (col. 5). The theoretical option prices (col. 7) and delta values (col. 2) are computed via (2.19) and (2.20) respectively. We may note that the value of the delta-derived replicating portfolio (col. 6) tracks very well the option prices (col. 7).

Table 1: One realization of the delta hedging strategy (the barrier is breached on the first rebalance of the fourth day).

Asset price	Delta	Shares purchased	Cost	Bank	Portfolio	Option price
1	2	3	4	5	6	7
102.598	–0.109	–0.069	–7.057	–12.540	1.374	1.330
102.637	–0.112	–0.003	–0.300	–12.840	1.370	1.372
102.544	–0.111	0.001	0.083	–12.757	1.380	1.432
105.763	–0.199	–0.088	–9.325	–22.082	1.023	0.974
104.783	–0.182	0.017	1.777	–20.305	1.218	1.205
104.775	–0.188	–0.006	–0.613	–20.917	1.220	1.258
106.538	–0.241	–0.053	–5.692	–26.609	0.888	0.925
106.826	–0.258	–0.017	–1.804	–28.414	0.819	0.895
109.586	–0.308	–0.049	–5.395	–33.809	0.106	0.127
111.663	–0.313	–0.005	–0.602	–34.411	–0.533	–0.533

A novel definition of relative standard deviation of the hedging cost, which we denote σ_{rel}, is established. It measures correctly the exposure of the practitioner to risk (enforced by the strategy), or else, the efficiency of risk absorbtion assigned to the strategy. The measure serves aiding the practitioner to adjust the number of rebalances during the option's lifetime. Curiously, the notion that is present in the literature is different and looses the correct physical interpretation. Hence, we set

$$\sigma_{rel} = \frac{\sigma_{delta}}{\sigma_{no}}, \tag{3.2}$$

where $\sigma_{delta} = E[(P_{\bar{T}} - H_{\bar{T}})^2]$, $\sigma_{no} = E[H_0 - H_{\bar{T}})^2]$, $P_{\bar{T}}$ is the value of the delta hedging portfolio (which depends on the number of rebalances per day) and $\bar{T}$ is either the expiration or the knock out time. Whence, σ_{delta} assigns the standard deviation of the hedge cost of the delta hedging strategy and σ_{no} that of a strategy which is the nearest thing from doing nothing. Indeed, the "no" strategy is characterized by the fact that the short seller hedges his position with the portfolio valued at H_0 totally invested in the money market account and do nothing more. In this case, a hedge *per se* does not exist in fact, as the dealer assumes 100% of the risk to settle his liability. So, σ_{rel} expresses the proportion of risk that must be assumed by the practitioner to that absorbed by the delta hedging strategy (the text ahead illustrates the matter). The expression $\frac{\sigma_{delta}}{H_0}$ – usually found in the literature – does not provide such physical argument.

Remark. The mean value of $P_{\bar{T}} - H_{\bar{T}}$ and $H_0 - H_{\bar{T}}$ are sufficiently small – as they should be – so they are disregarded in (3.2).

Table 2 gives the relative standard deviations, the kurtosis and the asymmetry of hedge costs parameterized by the number of daily rebalances.

Table 2: Numerical results.

Rebalances per day	σ_{rel}	σ_{delta}	σ_{no}	Hedging cost	Curtosis	Asymmetry
1	0.43	1.125	0.382	0.0103	5.134	0.067
3	0.27	0.660	0.409	0.0022	7.903	0.017
6	0.20	0.474	0.422	0.0012	10.924	0.009
9	0.17	0.388	0.438	0.0013	14.361	0.103
12	0.15	0.338	0.444	0.0006	16.455	0.097

Column 2 tells us that, performing three rebalances per day, the delta hedging strategy absorbs (or eliminates) 73% of the risk, while the practitioner remains exposed to (or must assume) 27% of the risk. This exposure reduces to 20% (i.e., the strategy copes with 80% of the risk) in the case of 6 rebalances per day, and is augmented to 43% (which means an elimination of 57% of the risk) in the case of 1 rebalance per day. Note that the "no" strategy is not sensitive to the changes of the number of rebalances per day – which is indeed consistent with the strategy (col. 4).

In periods of stability, the volatility will be small, so the values of σ_{rel} will decrease accordingly, in which case, one rebalance per day or less would suffice. We note that the hedging cost, which assigns the gains/losses average in the long run, is almost zero, as it should be. We also notice that the kurtosis is greater than 3 and the asymmetry is small, for all rebalancing schemes.

4 CONCLUSION

Via the risk neutral pricing technique (no PDEs were involved), we have obtained closed-form expressions for the exact prices and hedges of an up-and-out call option with a constant barrier. We have assumed an arbitrary square integrable deterministic function of time to model the volatility and a zero interest rate for the riskless asset. This model is consistent with the fact that, in practice, traders often work with some devised deterministic behaviors for the volatility, as a guidance for their decision-making. It is noteworthy that the standard (constant) barrier option associated with this model *does not* allow closed-form expressions.

The formulas provided recover verbatim standard results as particular cases, indicating that these are ready-to-use formulas leading to a minimum computational effort. Also, the results of the simulations were very good. The delta-derived replicating portfolio tracked very well the option prices. Underpinned by the novel definition of relative standard deviation, we established correctly the proportion of risk assumed by the practitioner to that absorbed by the delta hedging strategy, as a function of the number of rebalances per day.

RESUMO. As quatro últimas décadas registraram um considerável esforço de pesquisa na obtenção de formas fechadas para os preços livres de arbitragem de versões modificadas de opções européias assim como para as estratégias associadas, permitindo-se também que a dinâmica do ativo de risco subjacente exibisse parâmetros não necessariamente constantes. Nesse trabalho obtemos a forma fechada do preço e da estratégia associada considerando uma opção com barreira *up-and-out* com volatilidade representada por funções determinísticas arbitrárias. Particularizando-se a volatilidade para o caso constante, as expressões resgatam as fórmulas de Black & Scholes. Este fato sugere que o tempo de processamento computacional seja mínimo. Introduzimos também um novo critério de desvio padrão relativo para aferição da exposição do agente financeiro ao risco (induzido pela estratégia), posto que o critério usual encontrado na literatura não traz uma correta interpretação física. A nova medida vem orientar o agente financeiro no ajuste do número de intervenções no porfolio até a expiração da opção.

Palavras-chave: opção com barreira, preços livres de arbitragem, medida Martingale, mudança de escala.

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