TEMA Tend. Mat. Apl. Comput., **12**, No. 2 (2011), 91-99.
doi: 10.5540/tema.2011.012.02.0091

Mathematical Analysis of a Third-order Memristor-based Chua's Oscillator[1]

V.A. BOTTA P.,[2] C. NÉSPOLI,[3] M. MESSIAS[4], Departamento de Matemática, Estatística e Computação - DMEC, Faculdade de Ciências e Tecnologia, UNESP - Univ Estadual Paulista, 19060-900, Presidente Prudente, SP, Brasil.

Abstract. In this paper we present a detailed linear analysis of the equilibrium points stability of a memristor oscillator mathematical model, given by a three-dimensional 5-parameter piecewise-linear system of ordinary differential equations. We perform the linear analysis in the general case and present numerical simulations for some particular parameter values.

Keywords. Memristor oscillator, Chua oscillator, linear stability, bifurcation analysis.

1. Introduction

In 2008, a team of scientists of Hewlett-Packard Company announced the fabrication of a memristor, short for memory resistor, that is a passive nonlinear two-terminal circuit element that has a functional relationship between the time integrals of current and voltage (see e.g. [6]). The memristor is the fourth fundamental electronic element in addition to the resistor, inductor and capacitor. The existence of memristors was postulated theoretically by the scientist Leon Chua in 1971 [1].

According to Stan Willians of Hewlett-Packard Labs in Palo Alto, California, "a memristor is essentially a resistor with memory". So, for example, a computer created from memristive circuits can remember what has happened to it previously, and freeze that memory when the circuit is turned off [3]. Then, the memristor promises to be very useful in the fields of nanoelectronics and computer logic, for example. The announcement of the physical construction of the memristor has attracted worldwide attention due to its great potential in applications. In this way it is important the study of physical properties of the memristor in itself as well as its behavior when considered as part of a circuit with other electronic devices, like resistors, capacitors and inductors.

In Itoh and Chua [4], we can see the basic equations analysis of several types of nonlinear memristor oscillators, obtained by replacing Chuas's diode with memristors in some of the well studied Chua's circuits. Messias *et al.* [5] present a quite

[1]This work has been partially supported by Fundunesp (00863/09 - DFP) and by CNPq under the project 305204/2009-2. Paper presented at XXXIII CNMAC.
[2]botta@fct.unesp.br.
[3]cnespoli@fct.unesp.br.
[4]marcelo@fct.unesp.br.

Recebido em 08 Setembro 2010; Aceito em 07 Julho 2011.

complete bifurcation analysis of a third-order piecewise-linear canonical oscillator, which is among those derived in [4] and propose another model, obtained by replacing the piecewise-linear function (which represents the memductance) with a quadratic positive definite function.

In this paper we present a detailed linear bifurcation analysis of a third-order piecewise-linear memristor-based Chua oscillator, derived also by Itoh and Chua in [4], where the authors design a nonlinear oscillator by replacing the Chua's diode with an active two-terminal circuit consisting of a negative conductance and a memristor (or an active memristor). We numerically detect the existence of stable periodic orbits, which form periodic attractors for the studied model, for some parameter values.

Although obtained in the same paper [4], the systems studied here and in [5] are quite different, having only one equation in common. The system analyzed in [5] presents the relation among the three variables (v_1, i_3, φ), where v_1 represents the voltage, i_3 is the current and φ is the flux in the capacitor. Here, we have the relation among the variables (v, i), where v is the voltage and i is the current (for more details, see the obtainment of systems (65) and (105) in [4]).

2. A Third-order Memristor-based Chua's Oscillator

As described in Itoh and Chua [4], the memristor is a passive two-terminal electronic device described by a nonlinear constitutive relation between the device terminal voltage v and the terminal current i, given by

$$v = M(q)i, \quad \text{or} \quad i = W(\varphi)v. \tag{2.1}$$

The functions $M(q)$ and $W(\varphi)$, which are called *memresistance* and *memductance*, are defined by

$$M(q) = \frac{d\varphi(q)}{dq} \geq 0 \quad \text{and} \quad W(\varphi) = \frac{dq(\varphi)}{d\varphi} \geq 0, \tag{2.2}$$

and represent the slope of scalar functions $\varphi = \varphi(q)$ and $q = q(\varphi)$, respectively, named the *memristor constitutive relations*. In spite of the relations (2.1) simplicity, no one has been able to implement it in a physical device until the memristor fabrication announced in [6].

In [4] the authors give several mathematical models for memristor oscillators obtained by replacing Chua's diodes with memristors in some Chua's circuits. Here we consider the Van der Pol oscillator with Chua's diode, shown in Fig. 1 (see [4]). Replacing the Chua's diode with a two-terminal circuit consisting of a conductive and a flux-controlled memristor, we obtain the circuit shown in Fig. 2.

We can represent the dynamics of this circuit by the following system of first

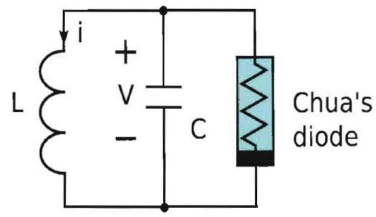

Figure 1: Van der Pol oscillator with Chua's diode.

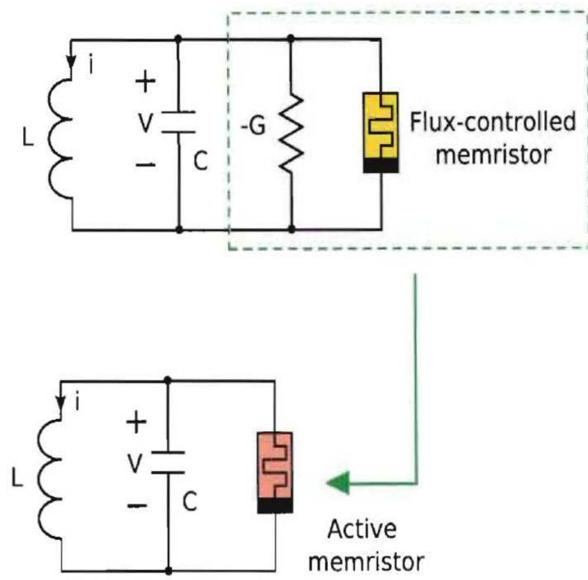

Figure 2: A third-order oscillator with a flux-controlled memristor and a negative conductance.

order ordinary differential equations

$$\begin{cases} C\dfrac{dv}{dt} = -i - W(\varphi)v + Gv \\ L\dfrac{di}{dt} = v \\ \dfrac{d\varphi}{dt} = v \end{cases}, \tag{2.3}$$

where

$$\begin{aligned} &W(\varphi) = \frac{dq(\varphi)}{d\varphi}, \\ &q(\varphi) = b\varphi + 0.5(a-b)(|\varphi+1| - |\varphi-1|). \end{aligned} \tag{2.4}$$

System (2.3) can be written as

$$\begin{cases} \dfrac{dx}{dt} = \alpha(-y - W(z)x + \gamma x) \\ \dfrac{dy}{dt} = \beta x \\ \dfrac{dz}{dt} = x \end{cases}, \tag{2.5}$$

where $x = v$, $y = i$, $z = \varphi$, $\alpha = \dfrac{1}{C}$, $\beta = \dfrac{1}{L}$, $\gamma = G$ and the piecewise-linear functions $q(z)$ and $W(z)$ are given by

$$\begin{aligned} q(z) &= bz + 0.5(a-b)(|z+1| - |z-1|), \\ W(z) &= \begin{cases} a, & if\ |z| < 1 \\ b, & if\ |z| > 1 \end{cases}, \end{aligned} \tag{2.6}$$

respectively, where $a, b > 0$.

In [4] the authors present numerical simulations of system (2.5) solutions for the particular parameter values $\alpha = 2,\ \beta = 1,\ \gamma = 0.3,\ a = 0.1$ and $b = 0.5$, for which they found two periodic attractors. In the next section we extend the analysis made there by performing a bifurcation study of system (2.5) when the parameters vary in the positive cone

$$\{(\alpha, \beta, \gamma, a, b) \in \mathbb{R}^5 \mid \alpha, \beta, \gamma, a, b > 0\}.$$

3. Linear Analysis

The equilibrium points of the system (2.5) are given by

$$E = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = 0 \text{ and } z \in \mathbb{R}\}.$$

Theorem 3.1. *The local normal stability of the equilibrium points* $(0, 0, z)$ *of system (2.5) with* $0 < a < b$ *is presented in Table 1, according to the positive parameters* α, β, γ, a *and* b, *where* $\tau = \alpha(\gamma - W(z))$, $D = \alpha\beta$ *and* $\Delta = \tau^2 - 4D$.

Proof. The Jacobian matrix J of system (2.5) at the equilibrium point $(0, 0, z)$ is given by

$$J = \begin{bmatrix} \alpha(\gamma - W(z)) & -\alpha & 0 \\ -\beta & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{3.1}$$

from which it follows that this equilibrium has the eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3}$ given by the solutions of the quadratic equation

$$\lambda^2 - \alpha\,(\gamma - W(z))\,\lambda + \alpha\beta = 0. \tag{3.2}$$

Assume that $\tau = \alpha(\gamma - W(z))$, $D = \alpha\beta$ and $\Delta = \tau^2 - 4D$, then the eigenvalues are given by

$$\lambda_1 = 0 \qquad \text{and} \qquad \lambda_{2,3} = \frac{\tau \pm \sqrt{\tau^2 - 4D}}{2}, \tag{3.3}$$

Table 1: Local normal stability of the equilibria $(0,0,z)$, $\mid z \mid \neq 1$, of system (2.5) according to the parameter values.

Conditions on τ	Conditions on Δ	Local stability of $(0,0,z)$ $\lvert z\rvert < 1$	Local stability of $(0,0,z)$ $\lvert z\rvert > 1$	Cases
$\tau < 0$ $\left(\dfrac{\gamma}{\alpha} < W(z)\right)$	$\Delta < 0$	stable foci	stable foci	(a)
	$\Delta = 0$	stable improper node	stable improper node	(b)
	$\Delta > 0$	stable node	stable node	(c)
$\tau > 0$ $\left(\dfrac{\gamma}{\alpha} > W(z)\right)$	$\Delta < 0$	unstable foci	unstable foci	(d)
	$\Delta = 0$	unstable improper node	unstable improper node	(e)
	$\Delta > 0$	unstable node	unstable node	(f)
τ changes signal with z $\left(a < \dfrac{\gamma}{\alpha} < b\right)$	$\Delta < 0$	unstable foci	stable foci	(g)
	$\Delta = 0$	unstable improper node	stable improper node	(h)
	$\Delta > 0$	unstable node	stable node	(i)

with the corresponding eigenvectors

$$v_1 = (0,0,1),\ v_2 = \Big(\frac{\tau - \sqrt{\Delta}}{2}, \frac{\tau^2 - \Delta}{4\alpha}, 1\Big)$$

and

$$v_3 = \Big(\frac{\tau + \sqrt{\Delta}}{2}, \frac{\tau^2 - \Delta}{4\alpha}, 1\Big).$$

Analyzing the possibilities for the eigenvalues (3.3) according to the relations between τ and D, which depend on the parameters $\alpha, \beta, \gamma, a, b$, we have the cases described in Table 1 for the (normal) local stability of the equilibria $(0,0,z)$, with $\mid z \mid < 1$ and $\mid z \mid > 1$.

It remains to verify the normal hyperbolicity of the equilibria $(0,0,z)$. In order to do that it is enough to calculate the equation of the plane π spanned by the eigenvectors v_2 and v_3 above. If λ_1 and λ_2 are real eigenvalues, then such plane has $n = v_2 \wedge v_3$ as its normal vector. Calculating the scalar product of n with the vector $(0,0,1)$ we obtain

$$\langle n, (0,0,1)\rangle = -\beta\sqrt{\Delta}, \tag{3.4}$$

which implies that the plane π is transversal do the z–axis, except in the case in which the parameters satisfy the condition $\Delta = 0$, because the vectors are linearly dependent in this case. Then by the Stable Manifold Theorem the invariant manifolds (stable and unstable) associated to the equilibria $(0,0,z)$ with $\mid z \mid \neq 1$ are tangent to the plane π, hence they are transversal to the z–axis. Therefore the equilibria are normally hyperbolic, except when $\Delta = 0$. In the case of complex eigenvalues we get the same type of result by considering the plane spanned by $Re(v_1)$ and $Im(v_1)$. This ends the proof of Theorem 3.1. $\square$

The case $0 < b < a$ can be treated in the same way.

From Theorem 3.1 we can obtain regions in the 5–dimensional parameter space for which all the equilibria $(0, 0, z)$, with $|z| \neq 1$, of system (2.5) are locally normally stable (cases (a), (b) and (c) of Table 1) and regions in which they are locally unstable (cases (d), (e) and (f)). On the other hand, there are regions in which, for a fixed set of the parameter values changes in the local stability of the equilibria may occur (cases (g), (h) and (i)). It is a type of bifurcation from a line of equilibria without varying the parameter values, i.e. a kind of "bifurcation without parameter", which has yet been described in the literature and mentioned before (see e.g. [2]). This type of bifurcation is an interesting dynamical phenomenon which has also been described for memristor models in [5].

The change in the local stability of an isolated equilibrium point of and ODE system is known to lead to important topological changes in its phase space, which are called *bifurcations* of the solutions. In the particular case in which the real part of a pair of complex conjugate eigenvalues cross transversely the imaginary axes as a parameter is varied, under certain conditions a limit cycle may be created, leading to oscillations of the solutions. It is the well known *Hopf bifurcation.* Observe that this change in the sign of the real part of the eigenvalues related to the equilibrium points $(0, 0, z)$ occurs for system (2.5), when the variable z crosses the values $z = \pm 1$ (see case (g) in Table 1), and it happens for a fixed set of parameter values.

In the next section we numerically found stable periodic orbits encircling the equilibrium $(0, 0, z)$ for $|z| < 1$ (the equilibrium point is locally unstable and the periodic orbit is stable). It is a type of Hopf bifurcation without parameter, which occur when the equilibrium point passes from stable for $|z| > 1$ to unstable for $|z| < 1$ (see also [5] for a detailed description of this phenomenon).

4. Numerical Simulations

In this section we present numerical simulations of system (2.5) for some parameter values.

Fig. 3 shows the local (normal) stability of the equilibria $(0, 0, z)$ where $\tau < 0$ and $\Delta < 0$ (case (a) of Table 1). In this case, the equilibria $(0, 0, z)$ have complex conjugate eigenvalues with negative real part for $|z| < 1$ and $|z| > 1$.

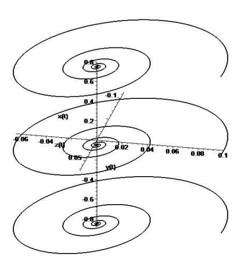

Figure 3: Solutions of system (2.5) with $\alpha = 4$, $\beta = 2$, $\gamma = 5$, $a = 5.2$ and $b = 10$, with $|z| < 1$. Initial conditions: (0.01, 0.1, -0.8), (0.01, 0.1, 0), (0.01, 0.1, 0.8). Integration time: [0,50].

Now we consider system (2.5) with the same parameter values used in [4], $\alpha = 2$, $\beta = 1$, $\gamma = 0.3$, $a = 0.1$ and $b = 0.5$ (case (g) of Table 1). We have changes in the local stability of the equilibria $(0, 0, z)$ as z crosses the values $z = \pm 1$. The solutions are shown in Figs. 4 and 5, including the periodic attractors.

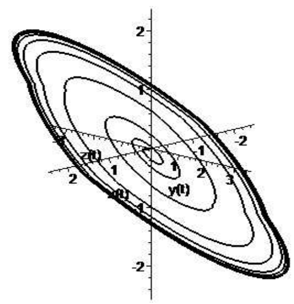

Figure 4: Solutions of system (2.5) with $\alpha = 2$, $\beta = 1$, $\gamma = 0.3$, $a = 0.1$ and $b = 0.5$, with $|z| < 1$. Initial condition: (0.1, 0.1, 0). Integration time: [0,50].

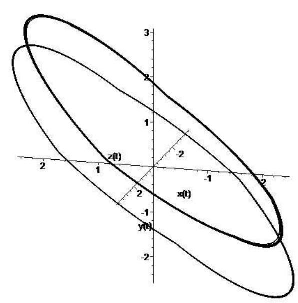

Figure 5: Solutions of system (2.5) with $\alpha = 2$, $\beta = 1$, $\gamma = 0.3$, $a = 0.1$ and $b = 0.5$, with $|z| < 1$. Initial conditions: (0.1, 0.1, 0), (0.1, 0.1, 0.9). Integration time: [50,100].

Fig. 6 shows the local (normal) instability of the equilibria $(0, 0, z)$ where $\tau > 0$ and $\Delta < 0$ (case (d) of Table 1), with $z \in \{-0.8, 0, 0.8\}$ and time of integration $t \in [0, 50]$. Fig. 7 shows the same solutions of Fig. 6 with time of integration $t \in [100, 120]$. We can see that globally the solutions tend towards stable periodic orbits around the $z-$axis.

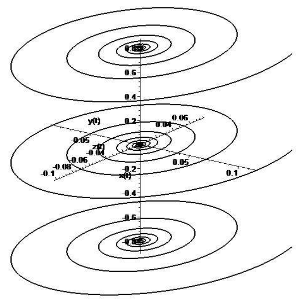

Figure 6: Solutions of system (2.5) with $\alpha = 1$, $\beta = 1$, $\gamma = 0.2$, $a = 0.02$ and $b = 2$, with $|z| < 1$. Initial conditions: (0.001, 0.001, -0.8), (0.001, 0.001, 0), (0.001, 0.001, 0.8). Integration time: [0,50].

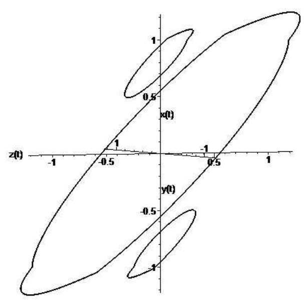

Figure 7: Solutions of system (2.5) with $\alpha = 1$, $\beta = 1$, $\gamma = 0.2$, $a = 0.02$ and $b = 2$, with $|z| < 1$. Initial conditions: (0.001, 0.001, -0.8), (0.001, 0.001, 0), (0.001, 0.001, 0.8). Integration time: [100,120].

Resumo. Neste trabalho apresentamos uma análise linear da estabilidade dos pontos fixos de um modelo matemático de um circuito elétrico envolvendo um memristor, dado por um sistema tri-dimensional de equações diferenciais ordinárias, com cinco parâmetros e uma função linear por partes. Apresentamos uma análise linear para o caso geral, envolvendo os cinco parâmetros, e apresentamos as simulações numéricas para alguns valores particulares dos parâmetros.

References

[1] L.O. Chua, Memristor – the missing circuit element, *IEEE Trans. Circuit Th.* **18** (1971), 507–519.

[2] B. Fiedler, S. Liebscher, J.C. Alexander, Generic Hopf bifurcation from lines of equilibria without parameters: I. Theory, *J. Differential Equations* **167** (2000), 16–35.

[3] M. Hopkin, Found: the missing circuit element, *Nature News*, April 2008.

[4] M. Itoh, L. O. Chua, Memristor oscillators, *Internat. J. Bifur. Chaos Appl. Sci. Engrg*, **18** (2008), 3183–3206.

[5] M. Messias, C. Nespoli, V.A. Botta, Hopf bifurcation from lines of equilibria without parameters in memristor oscillators, *Internat. J. Bifur. Chaos Appl. Sci. Engrg* **20** (2010) 437–450.

[6] D.B. Strukov, G.S. Snider, G.R. Stewart, R.S. Williams, The missing menristor found, *Nature* **453** (2008), 80–83.