On the Sizes of Maximal Independent Sets of Cylindrical Grid Graphs

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ABSTRACT. If a graph G has exactly t different sizes of maximal independent sets, G belongs to a collection called $\mathcal{M}_t$. For the Cartesian product of the graph P_n, the path of length n, and C_m, the cycle of length m, called cylindrical grid, we present a method to find maximal independent sets having different sizes and a lower bound on t, such that these graphs belong to $\mathcal{M}_t$.

Keywords: Well-covered graph, maximal independent set, Cartesian product.

1   INTRODUCTION

In [9] Plummer defines a graph to be *well-covered* if all its maximal independent sets have the same size. Generalizing this concept, Finbow, Hartnell, and Whitehead [4] define, for every $t \in \mathbb{N}$, the set $\mathcal{M}_t$ as the set of graphs that have maximal independent sets of exactly t different sizes. With this notation, $\mathcal{M}_1$ is precisely the set of well-covered graphs.

Well-covered graphs have been investigated from several different parameters. See the survey [10], for more details. Topp and Volkmann [11] proposed the following question about Cartesian products and well-covered graphs: "Do there exist non well-covered graphs whose Cartesian product is well-covered?". This question was partially answered by Fradkin [5] for some classes of triangle-free graphs and recently a negative answer was given by Hartnell and Rall [7], for arbitrary graph. For $G \in \mathcal{M}_t$ and $v \in V(G)$, Barbosa and Hartnell [2] determine the extreme values that r can assume where $G \backslash v$ belongs to $\mathcal{M}_r$. Additional properties of graphs in this class are given in [1]. Results on $\mathcal{M}_t$, for $t \geq 2$, related to graphs without small cycles are also given in [3, 4, 6].

The Cartesian product $P_n \square C_m$ is a *cylindrical grid graph*. Every maximal independent set in a graph is a dominating set, although the converse is not always true. In [8], methods to find the domination number of cylindrical grid graphs $P_n \square C_m$ with $m \geq 3$ and $n = 2, 3$, and 4 were proposed. Moreover, bounds on the domination numbers were found when $n = 5$ and $m \geq 3$. We present a method to find maximal independent sets having different sizes and a lower bound for t, such that $P_n \square C_m$ belongs to $\mathcal{M}_t$.

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Before we present our results and proofs, we summarize our notation.

We consider finite, simple, and undirected graphs. For a graph G, the vertex set and the edge set are denoted $V(G)$ and $E(G)$, respectively. For a vertex u of G, its neighbourhood is denoted $N_G(u)$ and its *closed neighbourhood* denoted $N_G[u]$ is the set $N_G(u) \cup \{u\}$. For a set U of vertices of G, let

$$N_G(U) = \bigcup_{u \in U} N_G(u) \setminus U \text{ and } N_G[U] = N_G(U) \cup U.$$

A set $D \subseteq V(G)$ of a graph G is a *dominating set* if every vertex $v \in V(G) \setminus D$ is adjacent to some vertex $u \in D$. The *domination number* of G is the cardinality of a smallest dominating set of the graph G and is usually denoted by $\gamma(G)$.

A set I of vertices of a graph G is *independent* if no two vertices in I are adjacent. An independent set I of G is *maximal* if every vertex u in $V(G) \setminus I$ has a neighbour in I. An independent set I of G is *maximum* if G has no independent set J with $|J| > |I|$. We denote $i(G)$ the cardinality of a smallest maximal independent set in G. We denote $\alpha(G)$ the cardinality of a maximum independent set in G and $ms(G)$ the set of all sizes of maximal independent sets in G. Hence, if G is well-covered, then $i(G) = \alpha(G)$ and $|ms(G)| = 1$.

For any two graphs G and H, the *Cartesian product* $G \square H$ is the graph with vertex set $V(G) \times V(H)$, such that two vertices (u_1, v_1) and (u_2, v_2) of $G \square H$ are adjacent whenever $v_1 = v_2$ and $u_1 u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$. Here, the vertices of the path P_n or the cycle C_n are always denoted $0, 1, \ldots, n-1$. For the graph $P_n \square C_m$, we denote $(C_m)_i$ the graph $C_m \square \{i\}$, with $i \in V(P_n)$.

For naturals a, b, and c, with $c > b + 1$, we shall denote the set $\{i : a \leq i \leq b\} \cup \{c\}$ by $\{a, \ldots, b, c\}$.

2   RESULTS

In [8] Nandi, Parui, and Adhikari establish the following result regarding the domination number of Cartesian product of paths and cycles.

Theorem 1. *[8] For all $m \geq 3$,*

- $\gamma(P_2 \square C_m) = \begin{cases} \frac{m}{2}, & \text{if } m \equiv 0 \ (\text{mod } 4), \\ \left\lceil \frac{m+1}{2} \right\rceil, & \text{otherwise}; \end{cases}$

- $\gamma(P_3 \square C_m) = \left\lceil \frac{3m}{4} \right\rceil$;

- $\gamma(P_4 \square C_m) = \begin{cases} m + 1, & \text{for } m = 3, 5, 9, \\ m, & \text{otherwise}; \end{cases}$

- *for $m \geq 6$, $m + \left\lceil \frac{m}{5} \right\rceil \leq \gamma(P_5 \square C_m) \leq m + \left\lceil \frac{m}{4} \right\rceil$.*

One of our main results is the following. It will be proved in Section 2.2.

Theorem 2. *Let $n \geq 3$ and $m \geq 4$. If $G = P_n \square C_m$, then $G \in \mathcal{M}_t$ for some*

$$t \geq \begin{cases} n \left\lfloor \frac{m}{4} \right\rfloor, & \text{if } m \equiv 0, 2, 3 \ (\text{mod } 4), \\ n \left\lfloor \frac{m}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } m \equiv 1 \ (\text{mod } 4). \end{cases}$$

We begin showing the size of a maximum independent set in a graph $P_n \square C_m$.

Proposition 3. *For $n \geq 2$ and $m \geq 3$, $\alpha(P_n \square C_m) = n \left\lfloor \frac{m}{2} \right\rfloor$.*

Proof.   Since $\alpha(C_m) = \left\lfloor \frac{m}{2} \right\rfloor$ and we have n cycles C_m, we conclude that $\alpha(P_n \square C_m) \leq n \left\lfloor \frac{m}{2} \right\rfloor$. Now, we show that $\alpha(P_n \square C_m) \geq n \left\lfloor \frac{m}{2} \right\rfloor$ presenting a maximal independent set with this cardinality. Let I be the set of vertices (i, j) of $P_n \square C_m$ such that $i = 0, \ldots, n - 1$, $j = 0, \ldots, 2 \left\lfloor \frac{m}{2} \right\rfloor - 1$, and $i + j$ is an odd integer. The set I has cardinality $n \left\lfloor \frac{m}{2} \right\rfloor$ and it is a maximal independent set in $P_n \square C_m$. $\square$

In Theorem 7, we determine t such that $P_2 \square C_m$ belongs to $\mathcal{M}_t$. Before, we prove some preliminary results.

Lemma 4. *Let $m \geq 3$ and $G = P_2 \square C_m$. If I is a maximal independent set of G, then I has even cardinality.*

Proof.   Let I be a maximal independent set in G, $X_i = I \cap (C_m)_i$ and $x_i = |X_i|$, for $i = 0, 1$. We show that $x_0 = x_1$. By symmetry, consider the subgraph induced by $(C_m)_0 \backslash X_0$, denoted by H. The graph H has exactly x_0 disjoint connected paths. We denote them by $P(k)$, $k = 1, \ldots, x_0$. If $|V(P(k))| > 3$, for some $k \in \{1, \ldots, x_0\}$, $(C_m)_0 \backslash N_G[X_0]$ has some path with at least two vertices, which implies at least two consecutive vertices in $I \cap (C_m)_1$. Thus we may assume $|V(P(k))| \leq 3$, $k = 1, \ldots, x_0$.

Fix some $k \in \{1, \ldots, x_0\}$. Let $(v_1, v_2, \ldots, v_q)$, with $q = |V(P(k))|$, be one of the two orderings of the vertices of $P(k)$ such that adjacent vertices are consecutive. For each $j \in \{1, 2, \ldots, |V(P(k))|\}$, let v'_j be the neighbour of v_j in $(C_m)_1$. Note that exactly x_0 vertices in $(C_m)_1$ are in $N_G(X_0)$ and they are separated by at most three vertices. We have three cases for each $P(k)$. If $|V(P(k))| = 1$, v'_1 must belong to I since its neighbours in the same cycle are dominated by I. If $|V(P(k))| = 2$, v_1 and v_2 belong to $N_G(I)$ and exactly one of v'_1 and v'_2 must belong to I. If $|V(P(k))| = 3$, the vertices v_2, v'_1, v'_2, and v'_3 are not in $N_G(X_0)$. Hence v'_2 must be in I. Therefore, in all cases, for $k = 1, \ldots, x_0$, for every path $P(k)$, there is exactly one vertex in X_1, resulting in $|I \cap (C_m)_0| = |I \cap (C_m)_1|$. $\square$

Lemma 5. *Let $n \geq 2$, $m \geq 3$ and $G = P_n \square C_m$. If I is a maximal independent set in G, then*

$$|I \cap (C_m)_i| \geq \left\lceil \frac{m}{4} \right\rceil,$$

for $i \in \{0, n - 1\}$.

Proof. Let I be a maximal independent set in G, $X_i = I \cap (C_m)_i$ and $x_i = |X_i|$, for $i \in \{0, n-1\}$. Suppose $x_i < \lceil \frac{m}{4} \rceil$, for some $i \in \{0, n-1\}$. By symmetry, we may assume $i = 0$. Consider the graph $(C_m)_0$ and the subgraph H induced by $(C_m)_0 \backslash N_G[X_0]$. Since at most $3 \lceil \frac{m}{4} \rceil - 3$ vertices of $(C_m)_0$ are in $N_G[X_0]$, at least $\lceil \frac{m}{4} \rceil$ of those vertices are not in $N_G[X_0]$. Moreover, H has at most x_0 disjoint connected paths. Therefore, at least one of these paths has at least two vertices. Since I is maximal, two adjacent vertices in $(C_m)_1$ are in I. This contradicts the independence of I and completes the proof. □

Proposition 6. *For $m \geq 3$, $i(P_2 \square C_m) = 2 \lceil \frac{m}{4} \rceil$.*

Proof. Let $J = \{(0, i) : i \equiv 1 \ (\text{mod } 4)\} \cup \{(1, i) : i \equiv 3 \ (\text{mod } 4)\}$. We consider a set I with $I = J$, if $m \equiv 0 \ (\text{mod } 4)$, $I = J \cup \{(1, 0), (0, m-1)\}$, if $m \equiv 1 \ (\text{mod } 4)$, and $I = J \cup \{(1, 0)\}$, otherwise.

Note that I is a maximal independent set of $P_2 \square C_m$, and $|I| = 2 \lceil \frac{m}{4} \rceil$. Hence, $i(P_2 \square C_m) \leq 2 \lceil \frac{m}{4} \rceil$. By Lemma 5, $|I \cap (C_m)_0| + |I \cap (C_m)_1| \geq 2 \lceil \frac{m}{4} \rceil$ and the desired statement follows. □

Now, we can show the quantity of different sizes of maximal independent sets in $P_2 \square C_m$.

Theorem 7. *For $m \geq 3$ and $G = P_2 \square C_m$, $G \in \begin{cases} \mathcal{M}_{\lfloor \frac{m}{4} \rfloor}, & \text{if } m \equiv 1 \ (\text{mod } 4), \\ \mathcal{M}_{\lfloor \frac{m}{4} \rfloor + 1}, & \text{otherwise.} \end{cases}$*

Proof. By Proposition 3, $\alpha(G) = 2 \lfloor \frac{m}{2} \rfloor$, and by Proposition 6, $i(G) = 2 \lceil \frac{m}{4} \rceil$. We present maximal independent sets of G with every even cardinality between $2 \lceil \frac{m}{4} \rceil$ and $2 \lfloor \frac{m}{2} \rfloor$, in view of Lemma 4.

First, let $l = \begin{cases} m, & \text{if } m \equiv 0, 3 \ (\text{mod } 4), \\ m - 3, & \text{if } m \equiv 1 \ (\text{mod } 4), \\ m - 2, & \text{if } m \equiv 2 \ (\text{mod } 4), \end{cases}$

and

$$k = \begin{cases} \lfloor \frac{m}{4} \rfloor - 1, & \text{if } m \equiv 1 \ (\text{mod } 4), \\ \lfloor \frac{m}{4} \rfloor, & \text{otherwise.} \end{cases}$$

Let $I(0)' = \{(1, i), i \equiv 0 \ (\text{mod } 4)\}, i \leq l\} \cup \{(0, i), i \equiv 2 \ (\text{mod } 4), i \leq l\}$ and

$$I(0) = \begin{cases} I(0)', & \text{if } m \equiv 0, 3 \ (\text{mod } 4), \\ I(0)' \cup \{(0, m-1), (1, m-2)\}, & \text{if } m \equiv 1 \ (\text{mod } 4), \\ I(0)' \cup \{(0, m-1)\}, & \text{if } m \equiv 2 \ (\text{mod } 4). \end{cases}$$

Note that the set $I(0)$ is a maximal independent set in G and $|I(0)| = 2 \lceil \frac{m}{4} \rceil = i(G)$. For every $j \in \{1, \ldots, k\}$, let

$$I(j) = I(j-1) \backslash \{(0, 4j-2)\} \cup \{(0, 4j-3), (0, 4j-1), (1, 4j-2)\}.$$

For every $j \in \{0, \ldots, k\}$, $I(j)$ is a maximal independent set in G and $|I(j)| = 2\lceil \frac{m}{4} \rceil + 2j$.

Hence G has maximal independent sets of $k+1$ different sizes which results in $\lfloor \frac{m}{4} \rfloor$ if $1 \equiv m \pmod 4$, or $\lfloor \frac{m}{4} \rfloor + 1$, otherwise. This completes the proof. $\square$

By Theorem 2, when $n \geq 3$ and $m \geq 4$, $P_n \square C_m$ is not well-covered. The graph $P_2 \square C_5$ is well-covered and also $P_n \square C_3$ for $n \geq 2$, as we show in next proposition.

Proposition 8. *For $n \geq 2$, if $G = P_n \square C_3$, then G is well-covered.*

Proof. By Proposition 3, $\alpha(G) = n$. Let I be a maximal independent set in G. We show that $|I| = n$. For a contradiction, suppose $|I| < n$. Hence there is some $j \in \{0, \ldots, n-1\}$, with $I \cap (C_3)_j = \emptyset$. Let S denote the set $N_G(V((C_3)_j))$. Since I is maximal and from the structure of G, $|S \cap I| \geq 3$. But S induces one or two cycles C_3, which contradicts the independence of I. $\square$

Proposition 9. *Let $n \geq 3$, $m \geq 4$ and $G = P_n \square C_m$. If m is even, then there is no maximal independent set I in G such that $|I| = \frac{nm}{2} - 1$;*

Proof. By Proposition 3, $\alpha(G) = \frac{nm}{2}$. A maximum independent set of G has $\frac{m}{2}$ vertices of every cycle $(C_m)_i$, for $i = 0, \ldots, n-1$. Suppose that there exists a maximal independent set I of G of size $\frac{nm}{2} - 1$. Any independent set of G has at most $\frac{m}{2}$ vertices on any cycle $(C_m)_i$, for $i = 0, \ldots, n-1$. Hence there exists some $i \in \{0, \ldots, n-1\}$ such that $|(C_m)_i \cap I| = \frac{m}{2} - 1$. At least one between $(C_m)_{i-1}$ and $(C_m)_{i+1}$ exists. Let $j \in \{i-1, i+1\}$ such that $(C_m)_j$ exists. Set $(C_m)_j \cap I$ has size $\frac{m}{2}$. Therefore, either all vertices (j, k) with k even or all vertices (j, k) with k odd are in $(C_m)_j \cap I$. Adjust notation of the vertex labeling of C_m such that all vertices (j, k) with k even are in I. Therefore, all vertices (i, f) of $(C_m)_i \cap I$ will be such that f is odd. Considering $m \geq 4$, $\frac{m}{2} - 1$ is at least one, thus $(C_m)_i \cap I$ has at least one vertex (i, f) with f odd. Therefore, if both $(C_m)_{i-1}$ and $(C_m)_{i+1}$ exist, then on both of them all vertices $(i-1, k)$ and $(i+1, k)$ for k even are in I. Considering there is exactly $\frac{m}{2} - 1$ vertices in $(C_m)_i \cap I$, there is a vertex (i, f) with f odd without a neighbour in $(C_m)_i \cap I$. However, (i, f) is not a neighbour of any vertex in I. Thus, I is not maximal. $\square$

2.1   Two useful classes of graphs

Before we prove Theorem 2, we show how to construct recursively two useful classes of graphs, denoted H_r and F_s. Furthermore, we show the different sizes of maximal independent sets in these graphs.

Let the graph H_1 have the vertex set $V^{(1)} = \{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\}$ and the edge set $E^{(1)} = \{v_1^{(1)}v_2^{(1)}, v_2^{(1)}v_3^{(1)}\}$. For $r \geq 2$, let the graph H_r have the vertex set

$$\begin{cases} V^{(r)} = V^{(r-1)} \cup \{v_1^{(r)}\}, & \text{for } r \text{ even}, \\ V^{(r)} = V^{(r-1)} \cup \{v_1^{(r)}, v_2^{(r)}, v_3^{(r)}\}, & \text{for } r \text{ odd}, \end{cases}$$

and the edge set

$$\begin{cases} E^{(r)} = E^{(r-1)} \cup \{v_1^{(r)}v_2^{(r-1)}\}, & \text{for } r \text{ even}, \\ E^{(r)} = E^{(r-1)} \cup \{v_1^{(r)}v_2^{(r)}, v_2^{(r)}v_3^{(r)}, v_2^{(r)}v_1^{(r-1)}\}, & \text{for } r \text{ odd}. \end{cases}$$
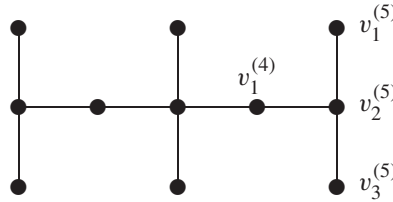
See Figure 1 for an example.

Figure 1: The graph H_5.

From the recursive construction, we can determine the sizes of maximal independent sets in H_r, as we prove in Proposition 10.

Proposition 10. *For $r \geq 2$, $H_r \in \mathcal{M}_r$. Furthermore,* $\mathrm{ms}(H_r) = \{\lceil \frac{r}{2} \rceil, \ldots, \lceil \frac{3r}{2} \rceil - 2, \lceil \frac{3r}{2} \rceil\}$.

Proof.   We prove the statement by induction on r. For $r = 2, 3, 4$, the result is trivial. Now let $r \geq 5$. We have to consider two cases. If r is even, by construction of H_r, it was obtained from H_{r-1} by the addition of the vertex $v_1^{(r)}$ and the edge $v_1^{(r)}v_2^{(r-1)}$. Since $v_1^{(r)}$ is a leaf, every maximal independent set in H_r contains either $v_1^{(r)}$ or $v_2^{(r-1)}$. Considering the first case, $H_r \setminus N_G[v_1^{(r)}]$ has a component isomorphic to H_{r-2} and two isolated vertices. By induction hypothesis, $H_{r-2} \in \mathcal{M}_{r-2}$ and $\mathrm{ms}(H_{r-2}) = \{\frac{r}{2} - 1, \ldots, \frac{3r}{2} - 5, \frac{3r}{2} - 3\}$. By adding the isolated vertices and $v_1^{(r)}$ we obtain maximal independent sets in H_r with sizes in $A = \{\frac{r}{2} + 2, \ldots, \frac{3r}{2} - 2, \frac{3r}{2}\}$. The graph $H_r \setminus N_G[v_2^{(r-1)}]$ is isomorphic to H_{r-3}. By induction hypothesis, $H_{r-3} \in \mathcal{M}_{r-3}$ and $\mathrm{ms}(H_{r-3}) = \{\frac{r}{2} - 1, \ldots, \frac{3r}{2} - 6, \frac{3r}{2} - 4\}$. By adding $v_2^{(r-1)}$ we obtain maximal independent sets in H_r with orders in $B = \{\frac{r}{2}, \ldots, \frac{3r}{2} - 5, \frac{3r}{2} - 3\}$. Since $A \cup B = \mathrm{ms}(H_r) = \{\frac{r}{2}, \ldots, \frac{3r}{2} - 2, \frac{3r}{2}\}$, the desired statement follows for r even, with $r \geq 5$.

If r is odd, by construction of H_r, it was obtained from H_{r-1} by the addition of the vertices $v_1^{(r)}, v_2^{(r)}$ and $v_3^{(r)}$, and the edges $v_1^{(r)}v_2^{(r)}, v_2^{(r)}v_3^{(r)}$ and $v_2^{(r)}v_1^{(r-1)}$. Since $v_1^{(r)}$ and $v_3^{(r)}$ have degree one, every maximal independent set in H_n contains either $v_1^{(r)}$ and $v_3^{(r)}$, or $v_2^{(r)}$. Considering the first case, $H_r \setminus N_G[\{v_1^{(r)}, v_3^{(r)}\}]$ is isomorphic to H_{r-1}. By induction hypothesis, $H_{r-1} \in \mathcal{M}_{r-1}$

and $\mathrm{ms}(H_{r-1}) = \{\frac{r-1}{2}, \ldots, \frac{3r-1}{2} - 3, \frac{3r-1}{2} - 1\}$. By adding $v_1^{(r)}$ and $v_3^{(r)}$ we have maximal independent sets in H_n with sizes in $C = \{\frac{r+1}{2} + 1, \ldots, \frac{3r+1}{2} - 2, \frac{3r+1}{2}\}$. The graph $H_r \setminus N_G[v_2^{(r)}]$ is isomorphic to H_{r-2}. By induction hypothesis, $H_{r-2} \in \mathcal{M}_{r-2}$ and $\mathrm{ms}(H_{r-2}) = \{\frac{r-1}{2}, \ldots, \frac{3r+1}{2} - 5, \frac{3r+1}{2} - 3\}$. By adding the vertex $v_2^{(r)}$ we obtain maximal independent sets in H_r with sizes in $D = \{\frac{r+1}{2}, \ldots, \frac{3r+1}{2} - 4, \frac{3r+1}{2} - 2\}$. Since $C \cup D = \mathrm{ms}(H_r) = \{\frac{r+1}{2}, \ldots, \frac{3r+1}{2} - 2, \frac{3r+1}{2}\}$, the desired statement follows for r odd, with $r \geq 5$. This completes the proof. $\qquad\square$

Proposition 11. *Let r be a fixed positive integer with $r \geq 3$ and $t \geq 1$. If G is a graph with t components, each one isomorphic to H_r, then $G \in \mathcal{M}_{rt}$ and,*

$$\mathrm{ms}(G) = \left\{ t \left\lceil \frac{r}{2} \right\rceil, \ldots, t \left(\left\lceil \frac{r}{2} \right\rceil + r \right) - 2, t \left(\left\lceil \frac{r}{2} \right\rceil + r \right) \right\}.$$

Proof. For $t = 1$ the result is trivial in view of Proposition 10. We consider $t \geq 2$. Let L_j, for $j \in \{1, \ldots, t\}$, be the components of G. Let $I(j)$ be a maximal independent set in the subgraph L_j. For every $j \in \{1, \ldots, t\}$, $\bigcup_{j=1}^{t} I(j)$ is a maximal independent set of G. By Proposition 10, $|I(j)| \in \{\lceil \frac{r}{2} \rceil, \ldots, \lceil \frac{r}{2} \rceil + r - 2, \lceil \frac{r}{2} \rceil + r\}$. Thus, all values on the interval $[\lceil \frac{r}{2} \rceil, \lceil \frac{r}{2} \rceil + r - 2]$ are present in $\mathrm{ms}(L_j)$. Therefore, it is possible to obtain a maximal independent set I of G with $|I| = x$ for all x in the interval $[t \lceil \frac{r}{2} \rceil, t(\lceil \frac{r}{2} \rceil + r - 2)]$, through the combination of maximal independent sets of graphs L_j. We are still missing all integers on the interval $[t(\lceil \frac{r}{2} \rceil + r - 2) + 1, t(\lceil \frac{r}{2} \rceil + r) - 2]$ and $t(\lceil \frac{r}{2} \rceil + r)$ for $\mathrm{ms}(G)$. For any integer $k \in \{0, \ldots, t\}$, it is possible to combine k maximal independent sets of $L_{j'}$, $j' = 0, \ldots, k$, of size $\lceil \frac{r}{2} \rceil + r - 2$ with $t - k$ maximal independent sets of the remaining $L_{j''}$, $j'' = k + 1, \ldots, t$, with size $\lceil \frac{r}{2} \rceil + r$, resulting in a maximal independent set of G with size $t(\lceil \frac{r}{2} \rceil + r) - 2k$. Thus, all integers on the set $\{t(\lceil \frac{r}{2} \rceil + r) - 2k : 0 \leq k \leq t\}$ are also on the set $\mathrm{ms}(G)$. For $r \geq 3$, both $(\lceil \frac{r}{2} \rceil + r - 2)$ and $(\lceil \frac{r}{2} \rceil + r - 3)$ belong to $\mathrm{ms}(L_j)$. So, by taking one maximal independent set of L_1 with size $\lceil \frac{r}{2} \rceil + r - 3$, $k - 1$ maximal independent sets of $L_{j'}$, $j' = 2, \ldots, k$, of size $\lceil \frac{r}{2} \rceil + r - 2$, and $t - k$ maximal independent sets of $L_{j''}$, $j'' = k + 1, \ldots, t$, with size $\lceil \frac{r}{2} \rceil + r$, we obtain a maximal independent set of G with size $t(\lceil \frac{r}{2} \rceil + r) - 2k - 1$, for $k = 1, \ldots, t$. Therefore, $\mathrm{ms}(G) = \{t\lceil \frac{r}{2} \rceil, \ldots, t(\lceil \frac{r}{2} \rceil + r) - 2, t(\lceil \frac{r}{2} \rceil + r)\}$. $\qquad\square$

Now we show how to construct a graph F_s, recursively.

Let the graph F_1 have the vertex set $V^{(1)} = \{u_1^{(1)}, u_2^{(1)}\}$ and the edge set $E^{(1)} = \{u_1^{(1)} u_2^{(1)}\}$. For $s \geq 2$, let the graph F_s have the vertex set $V^{(s)} = V^{(s-1)} \cup \{u_1^{(s)}, u_2^{(s)}\}$ and the edge set

$$\begin{cases} E^{(s)} = E^{(s-1)} \cup \{u_1^{(s)} u_2^{(s-1)}, u_1^{(s)} u_2^{(s)}\}, & \text{for } s \text{ even,} \\ E^{(s)} = E^{(s-1)} \cup \{u_2^{(s)} u_2^{(s-1)}, u_1^{(s)} u_2^{(s)}\}, & \text{for } s \text{ odd.} \end{cases}$$

Proposition 12. *For $s \geq 1$, the graph $F_s \in \mathcal{M}_{\lceil \frac{s}{2} \rceil}$ and $\mathrm{ms}(F_s) = \{\lfloor \frac{s}{2} \rfloor + 1, \ldots, s\}$.*

Proof. By construction, F_s has $\lfloor \frac{s}{2} \rfloor + 1$ vertices of degree one. Moreover, no pair of such vertices share a neighbour. Therefore, $i(F_s) \geq \lfloor \frac{s}{2} \rfloor + 1$. For every $i = 1, \ldots, s$, at most one vertex of $\{u_1^{(i)}, u_2^{(i)}\}$ can be in a maximal independent set of F_s, thus $\alpha(F_s) \leq s$. We conclude the proof showing how to find in F_s maximal independent sets with sizes in the set $\{\lfloor \frac{s}{2} \rfloor + 1, \ldots, s\}$. Let $I(0)$ be the set containing, for $i = 0, \ldots, \lceil \frac{s}{2} \rceil - 1$, the vertices $u_2^{(2i+1)}$. If s is even, the vertex $u_2^{(s)}$ must be added to $I(0)$. Note that $I(0)$ is a maximal independent set in F_s. For every $j \in \{1, \ldots, \lceil \frac{s}{2} \rceil - 1\}$, let $I(j) = (I(j-1) \setminus u_2^{(2j-1)}) \cup \{u_1^{(2j-1)}, u_1^{(2j)}\}$.

For $j = 0, \ldots, \lceil \frac{s}{2} \rceil - 1$, $I(j)$ is a maximal independent set in F_s and $|I(j)| = \lfloor \frac{s}{2} \rfloor + j + 1$, which implies $ms(F_s) = \{\lfloor \frac{s}{2} \rfloor + 1, \ldots, s\}$. □

2.2 Proof of Theorem 2

Now we use the graphs H_n and F_n and their different sizes of maximal independent sets to show a lower bound on the number of possible sizes of maximal independent sets in $P_n \square C_m$, for $n \geq 3$ and $m \geq 4$. We restate Theorem 2:

Theorem 2. *Let $n \geq 3$ and $m \geq 4$. If $G = P_n \square C_m$, then $G \in \mathcal{M}_t$ for some*

$$
t \geq \begin{cases} n \lfloor \frac{m}{4} \rfloor, & \text{if } m \equiv 0, 2, 3 \ (\text{mod } 4), \\ n \lfloor \frac{m}{4} \rfloor - \lfloor \frac{n}{2} \rfloor, & \text{if } m \equiv 1 \ (\text{mod } 4). \end{cases}
$$

Proof. We consider four cases:

Case 1. $m \equiv 0 \ (\text{mod } 4)$.

Let I be a set $\{(k, i) : k \equiv 1 \ (mod \ 2), i \equiv 0 \ (mod \ 4)\}$. The set I is independent and it has size $\frac{m}{4} \lfloor \frac{n}{2} \rfloor$. Let the graph H arise from G by deleting all vertices in $N_G[I]$. The graph H has $\frac{m}{4}$ components, and each one is isomorphic to the graph H_n. See Figure 2 for an illustration.

By Proposition 11, $H \in \mathcal{M}_{n(\frac{m}{4})}$ and

$$
ms(H) = \left\{ \left(\frac{m}{4}\right) \lceil \frac{n}{2} \rceil, \ldots, \left(\frac{m}{4}\right)\left(\lceil \frac{n}{2} \rceil + n\right) - 2, \left(\frac{m}{4}\right)\left(\lceil \frac{n}{2} \rceil + n\right) \right\}.
$$

By adding the vertices of I, we obtain $\frac{nm}{4}$ different sizes of maximal independent sets in G, which are $\{\frac{nm}{4}, \ldots, \frac{nm}{2} - 2, \frac{nm}{2}\}$.

Case 2. $m \equiv 1 \ (\text{mod } 4)$.

Let I be a set $\{(k, i) : k \equiv 1 \ (mod \ 2), i < m-1, i \equiv 0 \ (mod \ 4)\} \cup \{(k, m-4) : k \equiv 0 \ (mod \ 4)\} \cup \{(k, m-1) : k \equiv 2 \ (mod \ 4)\}$. The set I is independent and it has size $((\frac{m-1}{4} \lfloor \frac{n}{2} \rfloor) + \lceil \frac{n}{2} \rceil)$. Let G' be the graph arise from G by deleting all vertices in $N_G[I]$. The graph G' has $\frac{m-1}{4}$ components.
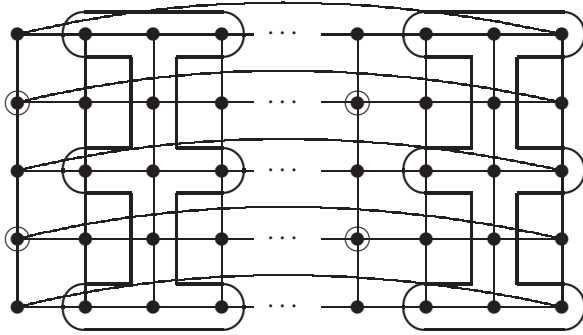
Figure 2: $G = P_5 \,\square\, C_m$, with $m \equiv 0 \pmod 4$. The independent set I is formed by the circled vertices and $G \setminus N_G[I]$ has $\frac{m}{4}$ components isomorphic to H_5.

Denote by H the graph whose components are the $\frac{m-5}{4}$ components of G' isomorphic to the graph H_n. The last component is isomorphic to the graph F_n. Denote it by H'. See Figure 3 for an illustration.

By Proposition 11,

$$\mathrm{m}s(H) = \left\{ \left(\frac{m-5}{4} \right) \left\lceil \frac{n}{2} \right\rceil, \ldots, \left(\frac{m-5}{4} \right)\left(\left\lceil \frac{n}{2} \right\rceil + n \right) - 2, \left(\frac{m-5}{4} \right)\left(\left\lceil \frac{n}{2} \right\rceil + n \right) \right\}.$$

By Proposition 12, $\mathrm{m}s(H') = \{ \lfloor \frac{n}{2} \rfloor + 1, \ldots, n \}$. The union of I with any maximal independent set of H and any independent set of H' is a maximal independent set of G.

Considering $n \geq 3$, we have that by Proposition 12, $ms(H')$ has at least two elements. Moreover, $ms(H)$ is an interval with one element missing. Then, $ms(G) = \{ n \lfloor \frac{m}{4} \rfloor + \lfloor \frac{n}{2} \rfloor + 1, \ldots, 2n \lfloor \frac{m}{4} \rfloor \}$ and $|ms(G)| = n \lfloor \frac{m}{4} \rfloor - \lfloor \frac{n}{2} \rfloor$.
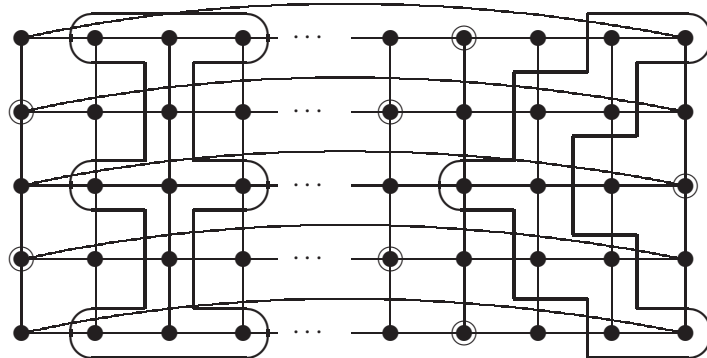

Figure 3: $G = P_5 \,\square\, C_m$, with $m \equiv 1 \pmod 4$. The independent set I is formed by the circled vertices and $G \setminus N_G[I]$ has $\frac{m-5}{4}$ components isomorphic to H_5 and another one isomorphic to F_5.

Case 3. $m \equiv 2 \pmod 4$.

Let I be a set $\{(k, i) : k \equiv 1 \pmod 2, i \equiv 0 \pmod 4\} \cup \{(k, m - 1) : k \equiv 0 \pmod 2\}$. The set I is independent and it has size $((\frac{m+2}{4} \lfloor \frac{n}{2} \rfloor) + \lceil \frac{n}{2} \rceil)$. Let the graph H arise from G by deleting all vertices in $N_G[I]$. The graph H has $\frac{m-2}{4}$ components, each of which is isomorphic to the graph H_n. See Figure 4(a) for an illustration. By Proposition 11,

$$\mathrm{ms}(H) = \left\{ \left(\frac{m-2}{4} \right) \left\lceil \frac{n}{2} \right\rceil, \ldots, \left(\frac{m-2}{4} \right) \left(\left\lceil \frac{n}{2} \right\rceil + n \right) - 2, \left(\frac{m-2}{4} \right) \left(\left\lceil \frac{n}{2} \right\rceil + n \right) \right\}.$$

By adding the vertices of I, we obtain $n \lfloor \frac{m}{4} \rfloor$ different sizes of maximal independent sets in G, which are $\{n \lceil \frac{m}{4} \rceil, \ldots, \frac{nm}{2} - 2, \frac{nm}{2}\}$.

(a)

(b)
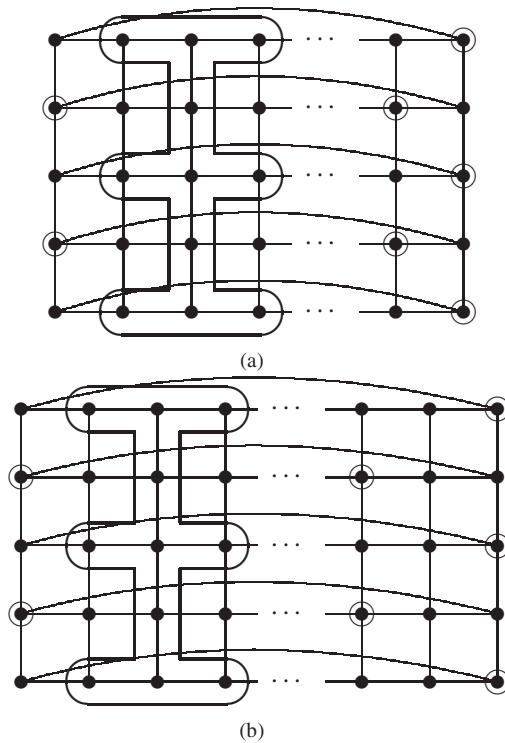
Figure 4: $G = P_5 \square C_m$, with $m \equiv 2 \pmod 4$ (a) and $m \equiv 3 \pmod 4$ (b). The independent set I is formed by the circled vertices and $G \setminus N_G[I]$ has $\lfloor \frac{m}{4} \rfloor$ components isomorphic to H_5.

Case 4. $m \equiv 3 \pmod 4$.

Let I be a set $\{(k, i) : k \equiv 1 \pmod 2, i \equiv 0 \pmod 4\} \cup \{(k, m - 1) : k \equiv 0 \pmod 2\}$. The set I is independent and it has size $((\frac{m+1}{4} \lfloor \frac{n}{2} \rfloor) + \lceil \frac{n}{2} \rceil)$. Let the graph H arise from G by deleting all

vertices in $N_G[I]$. The graph H has $\frac{m-3}{4}$ components, each of which is isomorphic to the graph H_n. See Figure 4(b) for an illustration. By Proposition 11,

$$\mathrm{ms}(H) = \left\{ \left(\frac{m-3}{4}\right)\left\lceil\frac{n}{2}\right\rceil, \ldots, \left(\frac{m-3}{4}\right)\left(\left\lceil\frac{n}{2}\right\rceil + n\right) - 2, \left(\frac{m-3}{4}\right)\left(\left\lceil\frac{n}{2}\right\rceil + n\right) \right\}.$$

By adding the vertices of I, we obtain $n\left\lfloor\frac{m}{4}\right\rfloor$ different sizes of maximal independent sets in G, which are $\{n\left\lceil\frac{m}{4}\right\rceil, \ldots, n\left\lfloor\frac{m}{2}\right\rfloor - 2, n\left\lfloor\frac{m}{2}\right\rfloor\}$. $\qquad\square$

3   CONCLUDING REMARKS

Since a vertex in a cylindrical grid graph G can dominate at most 5 vertices, a lower bound for $i(G)$ is $\left\lceil\frac{nm}{5}\right\rceil$, which gives us an upper bound $n\lfloor\frac{m}{2}\rfloor - \lceil\frac{nm}{5}\rceil + 1$ for the quantity of different sizes of maximal independent sets in G. Furthermore, the bound in Theorem 2 is sharp when m is a multiple of 4 and n is a small value. The technique we have applied may be useful to find maximal independent sets of different sizes in other graph classes or even in the study of other problems in Graph Theory.

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RESUMO. Se um grafo G tem exatamente t tamanhos diferentes de conjuntos independentes *maximais*, G pertence a uma coleção chamada $\mathcal{M}_t$. Para o produto Cartesiano do grafo P_n, o caminho de tamanho n, e C_m, o ciclo de tamanho m, chamado grade cilíndrica, apresentamos um método para encontrar conjuntos independentes *maximais* com diferentes tamanhos e um limite inferior para t, tal que estes grafos *pertençam* à $\mathcal{M}_t$.

Palavras-chave: Grafo bem-coberto, conjunto independente *maximal*, produto Cartesiano.

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